Continuous time simple random walk on a graph $\Gamma = (G, E)$.
Let $G$ be an infinite (connected) graph. For $x, y \in G$ let

$$\mu_{xy} = \mu_{yx} = \begin{cases} 
1 & \text{if } \{x, y\} \text{ is an edge}, \\
0 & \text{otherwise}.
\end{cases}$$

Define the vertex degree $\mu_x = \mu(x) = \sum_y \mu_{xy}$.
Assume $\mu(x) < \infty$ for all $x \in G$, and extend $\mu$ to a measure on $G$. Set $V(x, r) = \mu(B(x, r))$.

Dirichlet form:

$$\mathcal{E}(f, f) = \frac{1}{2} \sum_{x,y} \left( f(x) - f(y) \right)^2 \mu_{xy}.$$ 

The CTSRW on $\Gamma$ is the process

$$Y = (Y_t, t \in [0, \infty), P^x, x \in G)$$

associated with the Dirichlet form $(\mathcal{E}, L^2(G, \mu))$.

$Y$ waits at a point $x$ for an exponential time with mean 1, then moves to $y \sim x$ with probability $\mu_{xy}/\mu_x$. 
Heat kernel:

\[ q_t(x, y) = \frac{P^x(Y_t = y)}{\mu_y}. \]

Laplacian:

\[ \mathcal{L}f(x) = \frac{1}{\mu_x} \sum_y \mu_{xy}(f(y) - f(x)). \]

Heat equation:

\[ \frac{\partial}{\partial t} q_t(x_0, x) = \mathcal{L}q_t(x_0, x). \]

Discrete Gauss-Green:

\[ \mathcal{E}(f, g) = (-\mathcal{L}f, g), \text{ where } (\cdot, \cdot) \text{ is the inner product in } L^2(G, \mu). \]

Random walk on \( \mathcal{C}_\infty \) This is just the CTSRW on the graph \((\mathcal{C}_\infty(\omega), \mathcal{O}(\omega)|_{\mathcal{C}_\infty(\omega)})\). More precisely, we have two probability spaces:

1. \((\Omega, \mathbb{P}_p)\) - the space for the percolation process.
2. \(\Omega' (= D([0, \infty), \mathbb{Z}^d)\) for example\) - the space carrying the random walk. For each \(\omega \in \Omega\), and \(x \in \mathcal{C}_\infty(\omega)\) we then have the probability law \(P^x_\omega\) for \(Y\) started at \(x\). Write \(q_\omega^t(x, y)\) for the heat kernel of \(Y\).
Brief History

2. Grimmett, Kesten, Zhang, 1993: \( Y \) is recurrent if \( d = 2 \) and transient if \( d \geq 3 \).
   \[
   \mathbb{E}_pq^\omega_t(x, x) \geq ct^{-d/2}.
   \]
4. Benjamini, R. Lyons, Schramm 1999: bounded harmonic functions on \( C_\infty \) are constant. (Also results on general transitive graphs.)
   \[
   \mathbb{E}_p(q^\omega_t(x, x)|x \in C_\infty) \leq ct^{-d/2}(\log t)^{\delta_d}.
   \]
6. Mathieu & Remy (Ann Prob 2004): An isoperimetric inequality on \( C_\infty \) which implies that \( \mathbb{P}_p \) a.s. on \( \{\omega : x \in C_\infty(\omega)\} \),
   \[
   q^\omega_t(x, x) \leq ct^{-d/2}, \quad t \geq t_0(x, \omega).
   \]
8. MB (Ann Prob 2004): Gaussian upper and lower bounds on $q_t^\omega(x, y)$.
Theorem 3.1. Let $p > p_c$. For each $x \in \mathbb{Z}^d$ there exist r.v. $T_x(\omega)$ with

$$\mathbb{P}_p(T_x \geq n; x \in C_\infty) \leq c \exp(-n^{\varepsilon_d})$$

and (non-random) constants $c_i = c_i(d, p)$ such that for $x, y \in C_\infty(\omega)$:

$$q^\omega_t(x, y) \leq \frac{c_1}{t^{d/2}} e^{-c_2|x-y|^2/t} \quad \text{for } t \geq T_x(\omega) \lor |x - y|. \quad (GUB)$$

$$q^\omega_t(x, y) \geq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t} \quad \text{for } t \geq T_x(\omega) \lor |x - y|. \quad (GLB)$$

Note also

$$q^\omega_t(x, y) \geq \frac{c_3}{t^{d/2}} e^{-c_4|x-y|^2/t} \quad \text{for } t \geq T_x(\omega) \lor |x - y|^{3/2}. \quad (GLB')$$

Remarks.
1. CT$SRW$ on $\mathbb{Z}^d$ satisfies $(GB) = (GUB)+(GLB)$ with $T_x \equiv 1$.
2. We need $t \geq D = |x - y|$ for Gaussian behaviour. If $t \ll D$ then (even for $\mathbb{Z}^d$)

$$P^x(Y_t = y) \approx e^{-cD} P(\text{Poisson}(t) = D) \approx e^{-cD} e^{-cD \log(D/t)}.$$
3. There exist ‘long range’ bounds (Carne-Varopoulos, Davies) which give good upper bounds for any infinite graph. These bounds are:

\[ q_t(x, y) \leq c \exp(-c_1 d(x, y)^2 / t). \]

GUB follow if \( D^2 \geq at \log t \), or \( t < D^2 / \log D \). For when \( a \) is large enough

\[ e^{-c_1 D^2 / 2t} \leq e^{-c_1 a / 2 \log t} = t^{-ac_1 / 2} < t^{-d / 2} \]

4. The real zone of interest for these bounds is when

\[ 0 \leq D \leq (ct \log t)^{1/2}. \]

GLB' is \( D \leq t^{2/3} \) so contains this zone.

5. Applications of the bounds often need control on \( T_x(\omega) \) as well.

6. These bounds also hold (with minor modifications since \( C_\infty \) is bipartite) for the discrete time walk on \( C_\infty \).
A guide to how one proves this is:

**Theorem 3.2.** (Delmotte, (1999), simplified.) Let \( \Gamma \) be an infinite connected graph. The following are equivalent:

(a) \( \Gamma \) satisfies \( (V_d) \) and \( \text{(famPI)} \):

\[
    c_1 r^d \leq \mu(B(x, r)) \leq c_2 r^d \quad \text{for } x \in \mathcal{G}, r \geq 1.
\]

(b) \( q_t(x, y) \) satisfies GB with \( T_x \equiv 1 \),

(c) \( \Gamma \) satisfies Parabolic Harnack Inequality (PHI) and \( (V_d) \).

**Note:**

1. The more general version of this theorem, uses ‘volume doubling’ rather than \( (V_d) \).
2. \( \text{(famWkPI)} \) works as well as \( \text{(famPI)} \).
3. \( \Gamma \) satisfies \( \text{(famPI)} \) if for all \( B = B(x, r) \) one has the PI:

\[
    \int_B |f - \bar{f}_B|^2 d\mu \leq C_P r^2 \mathcal{E}_B(f, f) \quad \text{(PI)}
\]

4. The hard implication is \( \text{(a)} \Rightarrow \text{(b)}, \text{(c)} \). Delmotte proved \( \text{(a)} \Rightarrow \text{(c)} \) using Moser’s argument.
5. This extends to graphs earlier work on manifolds (Grigoryan, Saloff-Coste), metric spaces (Sturm).
(V_d) and (famWkPI) on \( \mathcal{C}_\infty \) 

Volume. We have

\[
\theta(p) = \mathbb{P}_p(|\mathcal{C}(x) = \infty) = \mathbb{P}_p(x \in \mathcal{C}_\infty).
\]

So by the ergodic theorem if \( Q_n \) is the box side \( n \) centre 0,

\[
\frac{|\mathcal{C}_\infty \cap Q_n|}{|Q_n|} \to \theta(p).
\]

Dividing a very big box into big boxes one deduces

\[
\mathbb{P}_p(c_1 R^d \leq |B_\omega(x, R)|) \geq 1 - e^{-cR^\delta}.
\]

Also \(|B_\omega(x, R)| \leq cR^d\) always.

Poincaré inequalities. Using the isoperimetric inequality one gets

\[
\mathbb{P}_p(\text{(wkPI) holds for } B_\omega(x, R)) \geq 1 - e^{-cR^\delta}.
\]

Hence by Borel-Cantelli for each \( x \) one has \( (V_d) \) and (wkPI) for \( B(x, R) \) for all \( R \geq R(x, \omega) \).