Divergence form operators on fractal-like domains

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Abstract. We consider elliptic operators $\mathcal{L}$ in divergence form on certain domains in $\mathbb{R}^d$ with fractal volume growth. The domains we look at are pre-Sierpinski carpets, which are derived from higher dimensional Sierpinski carpets. We prove a Harnack inequality for non-negative $\mathcal{L}$-harmonic functions on these domains and establish upper and lower bounds for the corresponding heat equation.

Keywords. Harnack inequality, divergence form, Sierpinski carpet, fractals, heat equation, elliptic operators, fundamental solution, diffusions

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1. Introduction.

The purpose of this paper is to consider divergence form operators on pre-Sierpinski carpets, to prove a Harnack inequality of Moser type, and to obtain estimates on the fundamental solution of the corresponding heat equation.

Let us begin by describing Sierpinski carpets, which are the class of fractal subsets of $\mathbb{R}^d$ formed by the following procedure. Let $d \geq 2$ and let $F_0 = [0,1]^d$. Let $l_F \geq 3$ be an integer and divide $F_0$ into $(l_F)^d$ equal subcubes. Next remove a symmetric pattern of subcubes from $F_0$ and call what remains $F_1$. Now repeat the procedure: divide each subcube that is contained in $F_1$ into $(l_F)^d$ equal parts, remove the same symmetric pattern from each as was done to obtain $F_1$ from $F_0$, and call what remains $F_2$. Continuing in this way we obtain a decreasing sequence of (closed) subsets of $[0,1]^d$. Let $\mathcal{F} = \cap_{n=0}^{\infty} F_n$; we call $\mathcal{F}$ a Sierpinski carpet or simply, a carpet. The standard Sierpinski carpet (see [Sie]) is the carpet for which $d = 2$, $l_F = 3$, and $F_1$ consists of $F_0$ minus the central square. If $d = 3$, $l_F = 3$, and $F_1$ consists of $F_0$ minus the 7 subcubes that do not share an edge with $F_0$, we obtain the Menger sponge; see [Man], p. 145 for a picture.

The domains we will consider are what are known as pre-carpets – see [O]. These are the sets $\mathcal{P} = \cup_{n=0}^{\infty} l_F^{-n} F_n$. (Here and throughout this paper we write $\lambda G = \{ \lambda x : x \in G \}$). Note that $\mathcal{P} \subset \mathbb{R}^d$, and that $\mathcal{P} \cap [0,l_F^{-n}]^d$ consists of $[0,l_F^{-n}]^d$ with a number of (possibly adjacent) cubical holes removed, of sides varying from 1 to $l_F^{-n+1}$. If $\Gamma$ is the interior of $\mathcal{P}$, then $\Gamma$ is a (non-empty) domain in $\mathbb{R}^d$ with a piecewise linear boundary. We may regard pre-carpets as idealized models of a region with obstacles of many different sizes. The set $\mathcal{P}$ is not a fractal, since the interior of $\mathcal{P}$ is a non-empty domain in $\mathbb{R}^d$. However, if we write $V(x,R)$ for the volume of the intersection of $\mathcal{P}$ with the Euclidean ball of radius $R$ centered at $x$, then $\mathcal{P}$ has ‘fractal volume growth’ in the sense that there exists $\alpha \in (1,d)$ such that

$$c_1 R^\alpha \leq V(x,R) \leq c_2 R^\alpha, \quad x \in \mathcal{P}, \quad R \geq 1.$$ 

Let $\mathcal{L}$ be the divergence form operator

$$\mathcal{L}f(x) = \sum_{i,j=1}^{d} \frac{\partial}{\partial x_i} \left( a_{ij} \frac{\partial}{\partial x_j} f \right)(x),$$

where the matrix $a_{ij}(x)$ is bounded, measurable, and symmetric for each $x$, and satisfies the uniform ellipticity condition

$$\lambda_1 |\xi|^2 \leq \sum_{i,j=1}^{d} \xi_i a_{ij}(x) \xi_j \leq \lambda_2 |\xi|^2, \quad x \in \mathcal{P}, \quad (1.1)$$

$$2.$$
with $0 < \lambda_1 < \lambda_2 < \infty$. We will assume the $a_{ij}$ are smooth, but our estimates will not depend on the smoothness of the $a_{ij}$. A function $f$ is $\mathcal{L}$-harmonic on a subdomain $D$ of $\mathcal{P}$ if $\mathcal{L}f = 0$ there and the conormal derivative of $f$ is 0 almost everywhere on $D \cap \partial \mathcal{P}$. For further information on diffusions with conormal reflection, see [PW].

![Diagram of a pre-carpet](image)

Figure 1: (Part of) a pre-carpet. The small squares have side length 1.

The main result of this paper is an elliptic Harnack inequality for $\mathcal{L}$-harmonic functions. For $x, y \in \mathcal{P}$ write $\gamma(x, y)$ for the (Euclidean) length of the shortest path in $\mathcal{P}$ connecting $x$ and $y$, and let $B_\gamma(x, r) = \{ y \in \mathcal{P} : \gamma(x, y) < r \}$.

**Theorem 1.1.** Let $x \in \mathcal{P}$, $R > 0$ and suppose $f$ is nonnegative and $\mathcal{L}$-harmonic in $B_\gamma(x, 2R)$. There exists $c_1$, not depending on $x$, $R$ or $f$, such that

$$f(x) \leq c_1 f(y) \quad \text{for } x, y \in B_\gamma(x, R).$$

A crucial point is that $c_1$ does not depend on $R$ for otherwise this result is an easy consequence of Moser’s Harnack inequality. In a previous paper [BB3] we proved Theorem 1.1 in the case $\mathcal{L} = \frac{1}{2} \Delta$, using a probabilistic coupling argument. This Harnack inequality was then the key step in obtaining bounds on the fundamental solution to the heat equation on $\mathcal{P}$:

$$\frac{\partial u}{\partial t} = \Delta u,$$
where \( u \) has Neumann boundary conditions on \( \partial \mathcal{P} \).

The arguments here do not replace those in [BB3]: we need various properties of the solutions to (1.3) (and the diffusion process \( W \) associated with them) to prove Theorem 1.1.

We should say a few words at this point about some of the difficulties in proving Theorem 1.1. Even a cursory glance at Moser’s method [M1] for proving a Harnack inequality as well as the method of Nash-Davies-Fabes-Stroock (see [FS]) shows that a Sobolev or Nash inequality is a crucial ingredient. Sobolev inequalities exist in the context of Sierpinski carpets; see [BB3], Section 7. However, both approaches rely on the existence of suitable ‘cut-off’ functions with bounded gradient. The results of Kusuoka [K] for the Sierpinski gasket suggest that such functions do not exist in the carpet case, and that, while they do exist in the pre-carpet case, scaling of the right order will not hold. Thus it is not clear how to use the above methods to prove Theorem 1.1 with \( c_1 \) independent of \( R \).

Our method is a variation of the Moser technique. The key step is to prove a weighted Sobolev inequality, where the \( L^p \) norm is with respect to an energy measure for the precarpet, rather than Lebesgue measure. We prove this inequality by first using Dirichlet form techniques to obtain a weighted Poincaré inequality, then to derive a weighted Nash inequality, and finally from this we obtain the weighted Sobolev inequality.

As in [BB3], once we have an elliptic Harnack inequality it is relatively straightforward to obtain bounds on the solutions of the associated heat equation \( \partial u / \partial t = \mathcal{L} u \) on \( \mathcal{P} \). The bounds are quite different from those for the heat equation on \( \mathbb{R}^d \). In the latter case, Aronson’s bounds tell us that the fundamental solution \( p(t, x, y) \) to \( \partial u / \partial t = \mathcal{L} u \) on \( \mathbb{R}^d \) is comparable to

\[
c_1 t^{-d/2} \exp(-c_2 |x - y|^2 / t).
\]

More precisely, for all \( x, y, t \) we have that \( p(t, x, y) \) is bounded above by \( c_1 t^{-d/2} \exp(-c_2 |x - y|^2 / t) \) and bounded below by \( c_3 t^{-d/2} \exp(-c_4 |x - y|^2 / t) \) for constants \( c_1, c_2, c_3, c_4 \). In contrast, for the heat equation on \( \mathcal{P} \), there exist constants \( d_w \) and \( d_s \) depending on \( \mathcal{P} \) such that for all \( t \geq 1 \) such that \( |x - y| \leq t \) the fundamental solution \( q(t, x, y) \) is comparable to

\[
c_1 t^{-d_s/2} \exp\left(-c_2 \left( \frac{|x - y|^{d_w}}{t} \right)^{1/(d_w-1)} \right);
\]

moreover \( d_w > 2 \). See Theorem 5.3 for a precise statement.

Our results are also of interest in light of recent research concerning necessary and sufficient conditions for a parabolic Harnack inequality to hold (see Grigor’yan [G] and Saloff-Coste [SC]) and also the discussion in [BB3]). Although both a Poincaré inequality and a volume doubling condition hold, the proofs of [G] and [SC] are not sufficient to prove that a parabolic or elliptic Harnack inequality hold on a Sierpinski carpet; their methods again require the use of suitable functions with bounded gradient.
In a series of papers Sturm has studied Harnack inequalities on metric spaces - see for example [St]. However, the hypotheses imposed rule out spaces with the kind of large scale fractal structure that pre-carpets have.

The layout of this paper is as follows. Section 2 introduces the notation we will use together with a few basic facts. Section 3 contains the proof of the weighted Sobolev inequality. We prove Theorem 1.1 in Section 4. The heat kernel bounds are derived in Section 5.

2. Notation and preliminaries.

We begin by setting up our notation. We use the letter $c$ with subscripts to denote constants which depend only on the dimension $d$ and the carpet $\mathcal{F}$. We renumber the constants for each lemma, proposition, theorem, and corollary. We use the notation $A \asymp B$ to mean $c_1 A \leq B \leq c_2 A$, where $c_i$ are as above.

Let $d \geq 2$, $F_0 = [0,1]^d$, and let $l_\mathcal{F} \in \mathbb{N}$, $l_\mathcal{F} \geq 3$ be fixed. For $n \in \mathbb{Z}$ let $\mathcal{S}_n$ be the collection of closed cubes of side length $l_\mathcal{F}^{-n}$ with vertices in $l_\mathcal{F}^{-n}\mathbb{Z}^d$. For $A \subseteq \mathbb{R}^d$, set

$$\mathcal{S}_n(A) = \{S : S \subset A, S \in \mathcal{S}_n\}.$$ 

For $S \in \mathcal{S}_n$, let $\Psi_S$ be the orientation preserving affine map which maps $F_0$ onto $S$.

We now define a decreasing sequence $(F_n)$ of closed subsets of $F_0$. Let $1 \leq m_\mathcal{F} < l_\mathcal{F}^d$ be an integer, and let $F_1$ be the union of $m_\mathcal{F}$ distinct elements of $\mathcal{S}_1(F_0)$. We impose the following conditions on $F_1$:

**Hypotheses 2.1.**

(H1) (Symmetry) $F_1$ is preserved by all the isometries of the unit cube $F_0$.

(H2) (Connectedness) The interior of $F_1$ is connected, and contains a path connecting the hyperplanes $\{x_1 = 0\}$ and $\{x_1 = 1\}$.

(H3) (Non-diagonality) Let $B$ be a cube in $F_0$ which is the union of $2^d$ distinct elements of $\mathcal{S}_1$. (So $B$ has side length $2l_\mathcal{F}^{-1}$). Then if the interior of $F_1 \cap B$ is non-empty, it is connected.

(H4) (Borders included) $F_1$ contains the line segment $\{x : 0 \leq x_1 \leq 1, x_2 = \ldots = x_d = 0\}$.

Of these, (H1) and (H2) are essential, while (H3) and (H4) could be weakened somewhat. See the discussion in [BB3].

We may think of $F_1$ as being derived from $F_0$ by removing the interiors of $l_\mathcal{F}^d - m_\mathcal{F}$ squares in $\mathcal{S}_1(F_0)$. Given $F_1$, $F_2$ is obtained by removing the same pattern from each of the squares in $\mathcal{S}_1(F_1)$. Iterating, we obtain a sequence $(F_n)$, where $F_n$ is the union of $m_\mathcal{F}^n$ squares in $\mathcal{S}_n(F_0)$. Formally, we define

$$F_{n+1} = \bigcup_{S \in \mathcal{S}_n(F_n)} \Psi_S(F_1) = \bigcup_{S \in \mathcal{S}_1(F_1)} \Psi_S(F_n), \quad n \geq 1.$$ 

We call the set $\mathcal{F} = \cap_{n=0}^{\infty} F_n$ a Sierpinski carpet.
Set
\[ \mathcal{P} = \bigcup_{r=0}^{\infty} l_{F_r}. \]

We call \( \mathcal{P} \) the \textit{pre-carpet} (see \cite{O}). We define the \textit{unbounded scaled pre-carpet} \( \mathcal{P}_N \) by
\[
\mathcal{P}_N = l_{F}^{-N} \mathcal{P} = \bigcup_{r=0}^{\infty} l_{F}^{-N} F_r, \quad N \geq 0.
\]

Until the end of Section 4 we will fix \( N \geq 0 \), and work on the scaled pre-carpet \( \mathcal{P}_N \). Any dependence of constants on \( N \) will be given explicitly.

We will require a certain amount of notation to describe various subsets of \( \mathcal{P}_N \). Let \( \mathcal{S}_n \) be the set of cubes in \( \mathbb{R}^d \) of side length \( 2l_{F}^{-n} \) which are unions of \( 2^d \) cubes in \( \mathcal{S}_n \). For \( x \in \mathcal{P}_N \) let \( Q(x) \) be the cube in \( \mathcal{S}_n \) with center closest to \( x \). (We use some procedure to break ties.) Set \( D_n(x) = Q(x) \cap \mathcal{P}_N \).

If \( x = (x_1, \ldots, x_d) \) is a point in \( \mathbb{R}^d \), write \( \|x\|_{l^\infty} = \max_{1 \leq i \leq d} |x_i| \). For \( x, y \in \mathcal{P}_N \) let \( d(x, y) \) denote the length of the shortest path (i.e., geodesic) in \( \mathcal{P}_N \) connecting \( x \) and \( y \), where the length of the path is measured in terms of the \( l^\infty \) norm. We have
\[
d(x, y) = \lim_{\delta \to 0} \left( \inf \left\{ \sum_{i=0}^{m} ||x_i - x_{i-1}||_{l^\infty} : x_0 = x, x_m = y, ||x_i - x_{i-1}||_{l^\infty} < \delta, x_i \in \mathcal{P}_N \right\} \right).
\]

We write \( B(x, r) = \{ y \in \mathcal{P}_N : d(x, y) < r \} \). Note that the boundary of \( B(x, r) \) is a finite union of flat surfaces orthogonal to the axes. We write \( \text{diam}(A) \) for the diameter of \( A \) in the metric \( d \), and \( \text{dist}(A, B) \) for the distance between the sets \( A \) and \( B \).

For \( A \subset \mathbb{R}^d \) we write \( A^0, \text{cl}(A), A^c \) for the usual interior, closure and complement of \( A \), respectively. Let \( D \) be a relatively open set in \( \mathcal{P}_N \) (in the metric \( d \)). We write \( \partial_{a} D \) for the relative boundary of \( D \) in \( \mathcal{P}_N \), and \( \partial_{c} D \) for the boundary of \( D \) in \( \mathbb{R}^d \). Set \( \partial_{r} D = \partial_{c} D - \partial_{a} D \). We use the subscripts \( a, c, r \) as mnemonics for ‘absorbing’, ‘everywhere’, and ‘reflecting’, respectively. If \( S \in \mathcal{S}_n \) for some \( n \in \mathbb{Z} \), and \( S^0 \cap \mathcal{P}_N \) is non-empty we call \( S \cap \mathcal{P}_N \) a special \( \mathcal{P}_N \) \textit{cube}. Note that if \( Q \) is any special \( \mathcal{P}_N \) cube of side length \( l_{F}^{-n} \) then \( Q \) is isomorphic to \( l_{F}^{-n} F_{n-n} \), where we write \( F_{-k} = [0,1]^d, k \geq 1 \). If \( B = B(x, r) \) then we define \( B^* = B(x, 2r) \). Let \( I = S \cap \mathcal{P}_N \) be a special \( \mathcal{P}_N \) cube. Write \( S' \) for the cube with side length 3 times that of \( S \) with the same center as \( S \) and faces parallel to those of \( S \). Let \( I^* = \text{cl}((S' \cap \mathcal{P}_N)^0) \).

Let \( Q \subset \mathbb{R}^d \) be a cube with edges parallel to the axes. We call any set of the form \( Q \cap \mathcal{P}_N \) a \( \mathcal{P}_N \)-\textit{cube}. Note that \( \mathcal{P}_N \) cubes do not have to be connected.

We define the resistance constant \( R_n \) by
\[
R_n^{-1} = \inf \{ \int_{l_{F}^{n} F_{n}} |\nabla f|^2 dx : f = 0 \text{ on } x_1 = 0, f = 1 \text{ on } x_1 = l_{F}^{n} \}.
\]
Thus $R_n$ is the resistance between two opposite faces of the set $l_F^n F_n$. It is known (see [BB3], [McG]) that there exists a constant $\rho_\mathcal{F}$ and constants $c_1, c_2$ such that

$$c_1 \rho_\mathcal{F}^n \leq R_n \leq c_2 \rho_\mathcal{F}^n.$$ 

Let $t_F = (m_\mathcal{F})(\rho_\mathcal{F})$. We define the fractal dimension, dimension of the walk, and spectral dimension of $\mathcal{F}$ by

$$d_f = \log m_\mathcal{F} / \log l_F,$$

$$d_w = \log t_F / \log l_F,$$

$$d_s = 2d_f / d_w = 2 \log m_\mathcal{F} / \log t_F.$$ 

d_f is the Hausdorff dimension (and also the packing dimension ) of $\mathcal{F}$. We remark (see [BB3, Remark 5.4]) that we have $d_w > 2$. We will also use

$$\zeta = \frac{l_F^2}{l_\mathcal{F}^2} = l_\mathcal{F}^{-d_w}. \quad (2.1)$$

Since $d_w > 2$ we have $\zeta < 1$. Let

$$\kappa = \frac{m_\mathcal{F}}{l_\mathcal{F}^d} = l_\mathcal{F}^{-d}. \quad (2.1)$$

This is the Lebesgue measure of $F_1$; we have $\kappa < 1$.

Let $|A|$ denote the Lebesgue measure of a Borel set $A$. If $Q$ is a special $\mathcal{P}_N$ cube of side length $s = l_\mathcal{F}^N$ then it is easy to check that

$$|Q| = \begin{cases} s^d, & \text{if } s \leq l_\mathcal{F}^{-N}, \\ \kappa^N s^d, & \text{if } s \geq l_\mathcal{F}^{-N}. \end{cases}$$

As the ball $B(x, r)$ contains a special $\mathcal{P}_N$ cube of side length $s \geq c_1 r$, and can be covered by $c_2$ or fewer special $\mathcal{P}_N$ cubes of side length $s \in [r, rl_\mathcal{F})$ we deduce that

$$|B(x, r)| \asymp \begin{cases} r^d, & \text{if } r \leq l_\mathcal{F}^{-N} \\ \kappa^N r^d, & \text{if } r \geq l_\mathcal{F}^{-N}. \end{cases} \quad (2.2)$$

Note that this implies that Lebesgue measure on $\mathcal{P}_N$ has the volume doubling property:

$$|B(x, 2r)| \leq c_1 |B(x, r)|, \quad r > 0.$$ 

Let $W_t$ be Brownian motion on $\mathcal{P}_N$ with normal reflection on the boundary of $\mathcal{P}_N$. Define $Y_t^N = W(\zeta^{-N} t)$. Then $Y_t^N$ is a process on $\mathcal{P}_N$ with generator $\frac{1}{2} \zeta^{-N} \Delta$, and Green function that is $\zeta^N$ times that of $W_t$. 

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If $D$ is a domain in $\mathcal{P}_N$ (so $D \subset \mathcal{P}_N$, $D$ is connected and relatively open in $\mathcal{P}_N$) write $u_D(x, y)$ for the Green function of $Y_N$ on $D$. Then $u_D$ is symmetric, continuous except on the diagonal $\{x = y\}$ and satisfies
\[ \frac{1}{2} \Delta u_D(x, y) = -\zeta^N \delta_x(y), \quad x, y \in D \]
in the distributional sense, where $\delta_x$ is point mass at $x$. If $D$ is suitably regular (such as a $\mathcal{P}_N$-cube or a ball) then we have $u_D(x, y) \to 0$ as $y \to \partial_4 D$; we extend $u_D$ to $\mathcal{P}_N \times \mathcal{P}_N$ by taking it to be zero off $D \times D$.

Let $D$ be a domain in $\mathcal{P}_N$, and $A \subset D$. Define
\[
U(x, A, D) = \int_A u_D(x, y) dy = \mathbb{E}^{x} \int_0^{\tau_D} 1_A(Y^N_t) ds;
\]
here $\tau_D = \inf\{s > 0 : Y^N_s \in D^c\}$. Note that $U$ is monotone in $A$ and $D$: if $A \subset A' \subset D \subset D'$ then
\[
U(x, A, D) \leq U(x, A', D') \leq U(x, A', D') \tag{2.3}
\]
Define the function
\[
\psi(r) = \begin{cases} \frac{r^d}{\zeta^N}, & \text{if } r \geq l_{\mathcal{F}}^{-N}, \\ \zeta^N r^2, & \text{if } r \leq l_{\mathcal{F}}^{-N}. \end{cases}
\]
Note that we can also write $\psi(r) = r^{d_w} \vee \zeta^N r^2$.

**Lemma 2.2.** Let $B$ be either a special $\mathcal{P}_N$ cube of side length $r$ or a ball $B(x_0, r)$. Then
(a) \quad $U(x, B, B) \leq c_1 \psi(r)$ for $x \in \mathcal{P}_N$,
(b) \quad $U(x, B^*, B^*) \geq c_2 \psi(r)$ for $x \in B$.
(c) \quad $U(x, B^*) \geq c_3 \psi(r)$ for $x \in B$.
(d) If $\frac{1}{2} t \leq s < t$ then
\[
U(x, B(x_0, s), B(x_0, t)) \geq c_4 \psi(t - s) \text{ for } x \in B(x_0, s).
\]

**Proof.** (a) and (b) follow from the estimates on hitting times of sets in [BB3], Proposition 5.5.
(c) We just do the case when $B = B(x_0, r)$; the result for special $\mathcal{P}_N$ cubes is very similar. Let $B_1 = B(x_0, r/2)$. Using the Markov property of $Y_N$ and writing $T_1 = \inf\{t \geq 0 : Y^N_t \in B_1\}$, $\tau = \inf\{t \geq 0 : Y^N_t \notin B^*\}$, we have for $x \in B$,
\[
U(x, B, B^*) \geq \mathbb{P}^x(T_1 < \tau) \inf_{y \in B_1} U(y, B, B^*)
\geq \mathbb{P}^x(T_1 < \tau) \inf_{y \in B_1} U(y, B(y, r/2), B(y, r/2))
\geq c_5 \mathbb{P}^x(T_1 < \tau) \psi(r/2).
\]
Here we used (2.3) and (b) to obtain the final line. It follows from the estimates on the transition density of $Y^N$ given in [BB3], Section 6, that there exists $c_6 > 0$ such that $\mathbb{P}^x(T_1 < \tau) > c_6$. Since $\psi(r/2) \geq c_7 \psi(r)$, (c) follows.
(d) Let \( y \in B(x_0, s) \). We can find a point \( z \) on the geodesic connecting \( y \) and \( x_0 \) such that \( y \in B(z, t - s) \subset B(x_0, s) \). Then \( B(z, 2(t - s)) \subset B(x_0, t) \), so, using (2.3),

\[
c_8 \psi(t - s) \leq U(y, B(z, t - s), B(z, 2(t - s)) \leq U(y, B(x_0, s), B(x_0, t)).
\]

\( \square \)

The following result generalizes Theorem 5.3 of [BB1]. Since there is an error in the proof of that result, we give details of the proof.

**Lemma 2.3.** There exist constants \( c_1, \beta > 0 \) such that if \( D \) is a \( \mathcal{P}_N \)-cube with side length less than \( l_N^{2/\beta} \) and \( A \) is a Borel subset of \( D \)

\[
|U(x, A, D) - U(y, A, D)| \leq c_1|x - y|^\beta, \quad x, y \in D. \tag{2.4}
\]

**Proof.** Let \( f = 1_A \) and write

\[
U_D f(x) = \int_D u_D(x, y) f(y) dy = U(x, A, D).
\]

For \( B \subset \mathcal{P}_N \) let

\[
\tau_B = \inf\{t \geq 0 : Y^N_t \in B^c\}
\]

be the first exit time of \( Y^N \) from \( B \).

By a proof almost identical with that of Theorem 5.2 of [BB1], there exist constants \( c_2, \beta_1 > 0 \) such that if \( D \) is a \( \mathcal{P}_N \)-cube with side length less than \( l_N^{2/\beta} \), then

\[
\mathbb{E}^x_{\tau_D} \leq c_2 d(x, \partial_a D)^{\beta_1}, \quad x \in D. \tag{2.5}
\]

Fix \( x, y \in D \). If \( 4d(x, y)^{1/2} > \text{dist}(x, \partial_a D) \), then

\[
|U_D f(x) - U_D f(y)| \leq \mathbb{E}^x_{\tau_D} + \mathbb{E}^y_{\tau_D} \leq c_3 d(x, y)^{\beta_1/2}
\]

by (2.5), and the theorem is proved in this case.

Now look at the case where \( 4d(x, y)^{1/2} \leq \text{dist}(x, \partial_a D) \). Let \( \delta = 4d(x, y)^{1/2} \) and let \( B = B(x, \delta) \). Then \( B \subset D \) and if \( z \in B \), then \( \tau_B \leq \tau_D \). So by the strong Markov property

\[
U_D f(z) = \mathbb{E}^z \int_0^{\tau_B} f(Y^N_t) dt + \mathbb{E}^z \int_{\tau_B}^{\tau_D} f(Y^N_t) dt \tag{2.6}
\]

\[
= \mathbb{E}^z \int_0^{\tau_B} f(Y^N_t) dt + \mathbb{E}^z U_D f(Y^N_{\tau_B}), \quad z \in B.
\]
The function $z \to \mathbb{E}^z U_D f(Y_{\tau_B}^N)$ is harmonic in $B$. The elliptic Harnack inequality for $Y_N$ (see Theorems 4.2 and 4.3 of [BB3]) implies there exist constants $c_4, \beta_2$ such that for $x', y' \in B(x, \delta/2)$

$$|\mathbb{E}^x U_D f(Y_{\tau_B}^N) - \mathbb{E}^{y'} U_D f(Y_{\tau_B}^N)| \leq c_4 \left( \frac{d(x', y')}{\delta} \right)^{\beta_2} \|U_D f\|_\infty. \quad (2.7)$$

By (2.5) we have

$$|U_D f(z)| \leq \|f\|_\infty \mathbb{E}^z \tau_D \leq c_5, \quad z \in D,$n

and

$$\mathbb{E}^z \int_0^{\tau_B} f(Y_t^N) dt \leq \|f\|_\infty \mathbb{E}^z \tau_B \leq c_6 \delta^{\beta_1}. \quad (2.8)$$

Combining (2.6), (2.7) with $x' = x$ and $y' = y$, and (2.8) with $z$ first equal to $x$ and then equal to $y$, we obtain our result in this case also.

There is a Poincaré inequality for $\mathcal{P}_N$ which may be stated in the following form.

**Lemma 2.4.** Let $B$ be either a special $\mathcal{P}_N$ cube of side length $r$ or a ball of radius $r$, and let $I = B$ or $I = B^*$. Suppose the gradient of $f$ is square integrable over $I$. Then, writing $f_I = |I|^{-1} \int_I f$,

$$\int_I |f - f_I|^2 \leq c_1 \psi(r) \zeta_i^{-N} \int_I |\nabla f|^2. \quad (2.9)$$

**Proof.** If $N = 0$ and $I = D_n(x)$ for some $n \leq 0$ then this is Proposition 7.10 of [BB3]. The case $N = 0$ and $I = D_n(x)$ with $n > 0$ is the usual Poincaré inequality in $\mathbb{R}^d$. The same argument as in [BB3] also proves this for special $\mathcal{P}_N$ cubes. The case with $N \geq 1$ follows easily by scaling.

To obtain (2.9) when $I$ is a ball we use the argument of Jerison [J], Section 5. We write $s(D_n(x))$ for the diameter (in the metric $d$) of the set $D_n(x)$: we have $s(D_n(x)) \asymp l_{\mathcal{F}} r^n$. Then it is quite straightforward to find a Whitney decomposition $\mathcal{F} = \{D_i = D_{n_i}(x_i), i \geq 1\}$ of $I$ with the following properties:

The sets $D_i^o$ are pairwise disjoint,

$I = \cup_i D_{n_i-2}(x_i),$

For each $y \in I$, $\# \{i : y \in D_{n_i-4}(x_i)\} \leq c_2$,

$l_{\mathcal{F}}^6 \leq \text{dist} (x_i, \partial_a I)/s(D_n(x_i)) \leq l_{\mathcal{F}}^2; \quad i \geq 1.$

Then, working with the sets $D_i$ rather than balls, the remainder of Jerison’s argument follows, with only minor changes, to give (2.9) for balls. \qed
3. Sobolev and other inequalities.

We continue to work on $\mathcal{P}_N$, and will mention explicitly any dependence on $N$ in our estimates.

For the remainder of this section we fix two $\mathcal{P}_N$-cubes $Q(h) \subset Q(k)$ such that $h = \text{diam}(Q(h)) \leq k = \text{diam}(Q(k)) \leq \frac{1}{2}$. Set

$$r(x) = U(x, Q(h), Q(k)), \quad x \in \mathcal{P}_N,$$

$$\gamma = 1 + \zeta^{-N} |\nabla r|^2.$$ 

Note that $r = 0$ off $Q(k)$, $r$ is strictly positive on $Q(h)$, and $\frac{1}{2} \Delta r = -\zeta^N 1_{Q(h)}$.

**Lemma 3.1.** (a) $r$ satisfies the bound

$$r(x) \leq c_1, \quad x \in \mathcal{P}_N.$$ 

(b) There exists $c_2$ such that

$$\int_{Q(k)} |\nabla r|^2 \leq c_2 \zeta^N |Q(h)| \sup_{Q(k)} |r|.$$ 

**Proof.** (a) is immediate from Lemma 2.2, and the fact that $Q(k)$ is contained in a ball of radius $\frac{1}{2}$. For (b) note that $r = 0$ on $\partial_{a} Q(k)$ and $\partial r / \partial n = 0$ on $\partial_r Q(k)$. So by Green’s first identity in the domain $Q(k)$,

$$\int_{Q(k)} |\nabla r|^2 = -\int_{Q(k)} r \Delta r = 2 \zeta^N \int_{Q(k)} r 1_{Q(h)}.$$ 

The result is now immediate. \qed

We will need the following elementary lemma.

**Lemma 3.2.** Let $x, y, z \geq 0$. If $x \leq c_1 (x^{1/2} z^{1/2} + y)$, then

$$x \leq 2c_1 y + 4c_1^2 z.$$ 

**Proof.** If $x < 2c_1 y$, we are done. If $x \geq 2c_1 y$, then $c_1 x^{1/2} z^{1/2} \geq x - c_1 y \geq x/2$, so $2c_1 z^{1/2} \geq x^{1/2}$. \qed

We begin by proving a weighted Poincaré inequality.
Proposition 3.3. Let $Q(h), Q(k), r$ be as in Lemma 3.1. Let $I \subset Q(k)$ be either a special $P_N$ cube of side length $s$ or a ball of radius $s$. Suppose $f$ and its gradient are square integrable over $I^*$. There exists $c_1 > 0$ such that

$$\int_I f^2|\nabla r|^2 \leq c_1 s^{2\beta} \left( \int_{I^*} |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_{I^*} f^2 \right).$$

Proof. Let $\varphi = U(\cdot, I, I^*)$, and write $\Phi_0 = \inf_I \varphi$, $\Phi_1 = \sup_{I^*} \varphi$. By Lemma 2.2 we have

$$c_2 \psi(s) \leq \Phi_0 \leq \Phi_1 \leq c_3 \psi(s).$$

Write

$$A = \int_{I^*} f^2 \varphi^2 |\nabla r|^2,$$
$$B = \int_{I^*} \varphi^2 |\nabla f|^2,$$
$$C = \int_{I^*} f^2,$$
$$D = \int_{I^*} f^2 |\nabla \varphi|^2,$$
$$E = \int_{I^*} |\nabla f|^2.$$

Then

$$\int_I f^2 |\nabla r|^2 \leq (\inf_I \varphi)^{-2} \int_I f^2 |\nabla r|^2 \varphi^2 \leq \Phi_0^{-2} A.$$

We begin by bounding $A$. Choose $x_0 \in I$. If $I^* \not\subset Q(k)$, set $\tilde{r} = r$. If $I^* \subset Q(k)$, set $\tilde{r} = r - r(x_0)$. In either case we see that there exists a point in $I^*$ at which $\tilde{r}$ is zero. Also, $\nabla \tilde{r} = \nabla r$, which is 0 off $Q(k)$. Set $R = \sup_{I^*} \tilde{r}$. By Lemma 2.3 $|\tilde{r}(x) - \tilde{r}(y)| \leq c_4 |x - y|^{\beta}$ if $x, y \in I^*$, and therefore $R \leq c_5 s^{\beta}$. We write

$$A = \int_{I^*} f^2 \varphi^2 |\nabla r|^2 = \int_{I^*} f^2 \varphi^2 |\nabla \tilde{r}|^2 = \int_{Q(k)} f^2 \varphi^2 |\nabla \tilde{r}|^2$$
$$= \frac{1}{2} \int_{Q(k)} f^2 \varphi^2 \Delta (\tilde{r}^2) - \int_{Q(k)} f^2 \varphi^2 \tilde{r} \Delta \tilde{r}, \quad (3.1)$$

where in the last line we used the identity $|\nabla u|^2 = \frac{1}{2} \Delta u^2 - u \Delta u$. 

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Now consider the first term on the right hand side of (3.1). If \( I^* \not\subset Q(k) \), then \( \partial(\tilde{r}^2)/\partial n = 2\tilde{r}(\partial \tilde{r}/\partial n) = 2r(\partial r/\partial n) \), which is 0 on \( \partial_a Q(k) \) and \( \partial_r Q(k) \). If \( I^* \subset Q(k) \), then \( \partial(\tilde{r}^2)/\partial n = 2\tilde{r}(\partial \tilde{r}/\partial n) = 2\tilde{r}(\partial r/\partial n) \) is 0 on \( \partial_r Q(k) \) and \( f^2\varphi^2 \) is 0 on \( \partial_a Q(k) \). So by Green’s first identity,

\[
\int_{Q(k)} f^2\varphi^2 \Delta(\tilde{r}^2) = - \int_{Q(k)} \nabla(f^2\varphi^2) \nabla(\tilde{r}^2).
\]

Thus

\[
A = - \frac{1}{2} \int_{Q(k)} \nabla(f^2\varphi^2) \nabla(\tilde{r}^2) - \int_{Q(k)} f^2\varphi^2 \Delta \tilde{r}
\leq \left| \int_{Q(k)} \nabla(f^2\varphi^2) \nabla(\tilde{r}^2) \right| + c_6 R \zeta^N \int_{I^*} f^2\varphi^2
\leq 4 \left( \int_{Q(k)} f^2\varphi(\nabla \varphi) \tilde{r} \nabla \tilde{r} \right) + 4 \left| \int_{Q(k)} f(\nabla f)(\varphi^2) \tilde{r} \nabla \tilde{r} \right| + c_6 R \zeta^N \Phi_1^2 \int_{I^*} f^2
\leq 4 \left( \int f^2\varphi^2 \tilde{r}^2 |\nabla \tilde{r}|^2 \right)^{1/2} \left( \int f^2 |\nabla \varphi|^2 \right)^{1/2}
+ 4 \left( \int f^2\varphi^2 \tilde{r}^2 |\nabla \tilde{r}|^2 \varphi^2 \right)^{1/2} \left( \int \varphi^2 |\nabla f|^2 \right)^{1/2} + c_6 R \zeta^N \Phi_1^2 C
\leq c_7 (R^2 A)^{1/2} (D^{1/2} + B^{1/2}) + c_6 R \zeta^N \Phi_1^2 C.
\]

As \( D^{1/2} + B^{1/2} \leq 2(B + D)^{1/2} \), by Lemma 3.2

\[
A \leq c_8 R^2 (B + D) + c_8 \zeta^N R \Phi_1^2 C.
\]

We now bound \( D \). We have

\[
D = \int_{I^*} f^2 |\nabla \varphi|^2 = \frac{1}{2} \int_{I^*} f^2 \Delta(\varphi^2) - \int_{I^*} f^2 \varphi \Delta \varphi.
\]

Since \( \partial(\varphi^2)/\partial n = 2\varphi(\partial \varphi)/\partial n \) is 0 on \( \partial_r I^* \) and 0 on \( \partial_a I^* \), by Green’s first identity

\[
\int_{I^*} f^2 \Delta(\varphi^2) = - \int_{I^*} \nabla(f^2) \nabla(\varphi^2).
\]

Since \( |\Delta \varphi| \) is bounded by \( 2\zeta^N \) on \( I^* \), then \( |\int_{I^*} f^2 \varphi \Delta \varphi| \leq c_9 \zeta^N \Phi_1 C \). So

\[
D \leq \left| \int_{I^*} \nabla(f^2) \nabla(\varphi^2) \right| + c_9 \zeta^N \Phi_1 C
= 4 \left| \int_{I^*} f(\nabla f)(\varphi^2) \nabla \varphi \right| + c_9 \zeta^N \Phi_1 C
\leq 4 \left( \int f^2 |\nabla \varphi|^2 \right)^{1/2} \left( \int \varphi^2 |\nabla f|^2 \right)^{1/2} + c_9 \zeta^N \Phi_1 C
\leq c_{10} (D^{1/2} B^{1/2} + \zeta^N \Phi_1 C).
\]
Using Lemma 3.2 again we conclude that
\[ D \leq c_{11}(B + \zeta^N \Phi_1 C). \]
Finally, as \( B \leq \Phi_2^2 E \), we deduce that
\[ A \leq c_{12} R^2 \Phi_1^2 E + c_{12} \zeta^N (R \Phi_1^2 + R^2 \Phi_1) C. \]
Since \( \psi(s) \leq c_{13} s^\beta \), and \( R \leq c_{14} s^\beta \) we have,
\[ \int_I f^2 |\nabla r|^2 \leq \Phi_0^{-2} A \leq c_{15} (\Phi_1/\Phi_0)^2 s^{2\beta} E + c_{15} \zeta^N (\Phi_1/\Phi_0)^2 s^{2\beta} \Phi_1^{-1} C. \]
Using the bounds above on \( \Phi_i \) the conclusion follows. \( \square \)

**Corollary 3.4.** Let \( f, I, \) and \( I^* \) be as in Proposition 3.3.
(a) Then if \( f_{I^*} = |I^*|^{-1} \int_{I^*} f \),
\[
\int_I (f - f_{I^*})^2 \gamma \leq c_1 \zeta^{-N} s^{2\beta} \int_{I^*} |\nabla f|^2. \tag{3.2}
\]
(b) Further,
\[
\int_I f^2 \gamma \leq c_2 \zeta^{-N} s^{2\beta} \int_{I^*} |\nabla f|^2 + |I|^{-1} \left( \int_I |f| \gamma \right)^2.
\]

**Proof.** Applying Proposition 3.3 to \( f - f_{I^*} \) we deduce
\[
\int_I (f - f_{I^*})^2 |\nabla r|^2 \leq c_3 s^{2\beta} \left( \int_{I^*} |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_{I^*} (f - f_{I^*})^2 \right). \tag{3.3}
\]
By the Poincaré inequality Lemma 2.4 we have
\[
\int_I (f - f_{I^*})^2 \leq \int_I (f - f_{I^*})^2 \leq c_4 \zeta^{-N} \psi(s) \int_{I^*} |\nabla f|^2. \tag{3.4}
\]
Substituting the second inequality of (3.4) into (3.3),
\[
\int_I (f - f_{I^*})^2 |\nabla r|^2 \leq c_5 s^{2\beta} \int_{I^*} |\nabla f|^2. \tag{3.5}
\]
Since \( \psi(s) \leq c_6 s^\beta \) we obtain (a) by adding (3.4) and (3.5).

(b) Now let \( b = \int_I f \gamma / \int_I \gamma \). Then
\[
\int_I f^2 \gamma = \int_I (f - b)^2 \gamma + b^2 \int_I \gamma
\]
\[
= \int_I (f - b)^2 \gamma + \left( \int_I \gamma \right)^{-1} \left( \int_I f \gamma \right)^2.
\]
\\[
\leq \int_I (f - f_{I^*})^2 \gamma + |I|^{-1} \left( \int_I f \gamma \right)^2. \tag{3.6}
\]
Combining (3.2) and (3.6) completes the proof. \( \square \)

We can obtain a sharper result if we just consider special \( P_N \) cubes.
**Corollary 3.5.** Let $I$ be a special $\mathcal{P}_N$ cube. Then

\[
\int_I f^2 |\nabla r|^2 \leq c_1 s^{2\beta} \left( \int_I |\nabla f|^2 + \zeta^N \psi(s)^{-1} \int_I f^2 \right),
\]

(3.7)

\[
\int_I f^2 \gamma \leq c_1 \zeta^{-N} s^{2\beta} \int_I |\nabla f|^2 + |I|^{-1} \left( \int_I |f| \gamma \right)^2.
\]

(3.8)

**Proof.** Note that the left-hand sides of (3.7) and (3.8) do not depend on the values of $f$ outside of $I$. Recall that $I^*$ is the union of the $(3^d$ or fewer) special $\mathcal{P}_N$-cubes of side length $s$ touching $I$; extend $f$ to a function $\tilde{f}$ on $I^*$ by reflection. Then

\[
\int_{I^*} \tilde{f}^2 \leq 3^d \int_I f^2, \quad \int_{I^*} |\nabla \tilde{f}|^2 \leq 3^d \int_I |\nabla f|^2,
\]

and (3.7) and (3.8) now follow from Proposition 3.3 and Corollary 3.4(b) for $\tilde{f}$. \qed

Next we proceed to a Nash inequality for special $\mathcal{P}_N$ cubes. Because the Laplacian is not a symmetric operator with respect to $\gamma$, we cannot use the method in $|\mathcal{S}C|$. 

**Proposition 3.6.** Let $J$ be a special $\mathcal{P}_N$ cube with side length $s \leq l^2_x$. Suppose the gradient of $f$ is square integrable over $J$ and $\int_J f^2 \gamma < \infty$. Then

\[
\int_J f^2 \gamma \leq c_1 \max \left( A^d/(2\beta+d), B^d/(2\beta+d) \right)
\]

where

\[
A = \zeta^{-N} \int_J |\nabla f|^2 + s^{-2\beta} \int_J f^2 \gamma, \quad B = \zeta^{-N} \left( \int_J |f| \gamma \right)^2.
\]

**Proof.** The result is trivial if $A = 0$, so we may assume $A > 0$. Let $t \in (0, s)$. We can find a covering of $J$ by special $\mathcal{P}_N$ cubes $I_i$ of side length between $t/l^2_x$ and $t$ such that $J = \cup I_i$, and the $I_i^o$ are disjoint. Note that $|I_i| \sim \kappa^N t^d_i \wedge t^d$. Set $\psi_0(t) = t^{-d_i} \vee (\kappa^N t^{-d})$. We apply Corollaries 3.4 and 3.5 and sum. So

\[
\int_J f^2 \gamma = \sum_i \int_{I_i} f^2 \gamma
\]

\[
\leq c_2 t^{2\beta} \zeta^{-N} \sum_i \int_{I_i} |\nabla f|^2 + c_2 \sum_i |I_i|^{-1} \left( \int_{I_i} |f| \gamma \right)^2
\]

\[
\leq c_2 t^{2\beta} c_3 \zeta^{-N} \int_J |\nabla f|^2 + c_2 \psi_0(t) \kappa^{-N} \left( \sum_i \int_{I_i} |f| \gamma \right)^2
\]

\[
\leq c_4 t^{2\beta} A + c_5 \psi_0(t) B.
\]

(3.9)
If \( t > s \) then (possibly adjusting the constant \( c_4 \)), the inequality (3.9) is trivial. If we now choose \( t_0 \) so that \( t_0^{2\beta} A = \psi_0(t_0)B \), then we have that
\[
  t_0 = \begin{cases} 
    (B/A)^{1/(2\beta+d)}t, & \text{if } t_0 \geq \frac{l}{T}, \\
    (\kappa^{N} B/A)^{1/(2\beta+d)}, & \text{if } t_0 \leq \frac{l}{T}.
  \end{cases}
\]

Now let \( t = t_0 \) and substitute in (3.9) to conclude the proof. \( \square \)

Next is a preliminary version of a weighted Sobolev inequality. Again the lack of symmetry of the Laplacian with respect to \( \gamma \) necessitates new methods.

**Proposition 3.7.** Let \( J \) be a special \( P_N \) cube with side length \( s \leq 1 \). Let \( f \) be as above. Then for any \( R \in (2, 2+2\beta/d) \) there exists \( c_1(R) < \infty \) such that
\[
  \left( \kappa^{-N} \int_J |f|^{R} \right)^{1/R} \leq c_1(R) \left[ \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-2\beta} \int_J f^2 \gamma \right]^{1/2}.
\]

**Proof.** Since \( |\nabla (f^+)| \leq |\nabla f| \) a.e. and \( |f| \leq f^+ + f^- \), it suffices to consider nonnegative \( f \). Write
\[
  A_0(f) = \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-2\beta} \int_J f^2 \gamma, \quad B_0(f) = \left( \kappa^{-N} \int_J |f| \gamma \right)^2.
\]

Multiplying \( f \) by \( A_0(f)^{-1/2} \), it is enough to prove
\[
  \kappa^{-N} \int_J |f|^{R} \gamma \leq c_1 \quad \text{if } A_0(f) = 1.
\]

Set
\[
  p_n = \kappa^{-N} \int_{\{ f \geq 2^n \} \cap J} \gamma.
\]

Then
\[
p_n \leq p_0 \leq \kappa^{-N} \int_{\{ f \geq 2^n \} \cap J} f^2 \gamma \leq \kappa^{-N} \int_J f^2 \gamma \leq s^{2\beta} A_0(f) \leq 1.
\]

Let \( f_n = (f \wedge 2^{n+1}) - (f \wedge 2^n) \); note that \( f_n \leq 2^n \), that \( f_n = 2^n \) on \( J \cap \{ f \geq 2^{n+1} \} \), and that \( f_n = 0 \) on \( \{ f < 2^n \} \). Therefore
\[
  \kappa^{-N} \int_J f_n \gamma = \kappa^{-N} \int_{\{ f \geq 2^n \} \cap J} f_n \gamma \leq \kappa^{-N} \int_{\{ f \geq 2^n \} \cap J} \gamma 2^n = 2^n p_n,
\]
while
\[
  \kappa^{-N} \int_J f_n^2 \gamma \geq \kappa^{-N} \int_{\{ f \geq 2^n \} \cap J} f_n^2 \gamma \geq \kappa^{-N} \int_{\{ f \geq 2^{n+1} \} \cap J} f_n^2 \gamma = 2^{n} p_{n+1}.
\]
Since \( \int_J f_n^2 \gamma \leq \int_J f^2 \gamma \) and \( \int_J |\nabla f_n|^2 \leq \int_J |\nabla f|^2 \), we have \( A_0(f_n) \leq A_0(f) \). So, from (3.13) we deduce \( p_n \leq 4 \cdot 2^{-2n} \). Applying Proposition 3.6 to \( f_n \) we have, using the fact that \( A_0(f) \leq 1 \),

\[
\kappa^{-N} \int_J f_n^2 \gamma \leq c_2 \max \left( B_0(f_n)^{2\beta/(2\beta+df)}, \kappa^N B_0(f_n)^{2\beta/(2\beta+d)} \right).
\]

Hence, we obtain

\[
2^{2n} p_{n+1} \leq c_3 \max \left( (2^n p_n)^{2\beta/(2\beta+df)}, (\kappa^N 2^n p_n)^{2\beta/(2\beta+d)} \right). \tag{3.14}
\]

Since \( 2^n p_n \leq 4 \) and \( df < d \), both the terms on the right hand side of (3.14) are dominated by \( c_4 (2^n p_n)^{2\beta/(2\beta+d)} \). Therefore

\[
p_{n+1} \leq c_4 (2^{-n})(2\beta+2d)/(2\beta+d)(p_n)^{2\beta/(2\beta+d)}.
\]

Elementary calculations now verify that \( p_n \leq a 2^{-n\theta} \) where \( \theta = 2(\beta + d)/d \) and \( a = c_5 \geq 1 \), is a constant depending only on \( c_4, \beta \) and \( d \).

Since

\[
\kappa^{-N} \int_J |f|^R \gamma \leq c_6 \sum_{n=0}^{\infty} 2^n R p_n,
\]

we deduce (3.11) (with a constant depending on \( R \)) for any \( R \in (2, 2 + 2\beta/d) \). \qed

We can modify slightly the final term in (3.10).

**Corollary 3.8.** Let \( J \) be a special \( \mathcal{P}_N \) cube of side \( s \leq \frac{l}{2} \). Let \( f \) and its gradient be square integrable over \( J \). Then for any \( R \in (2, 2 + 2\beta/d) \) there exists \( c_1(R) < \infty \) such that

\[
\left( \kappa^{-N} \int_J |f|^R \gamma \right)^{1/R} \leq c_1(R) \left[ \kappa^{-N} \int_J |\nabla f|^2 + \kappa^{-N} s^{-d_w} \int_J f^2 \right]^{1/2}. \tag{3.15}
\]

**Proof.** We have, using Corollary 3.5, and the fact that \( 2\beta \leq 2 < d_w \),

\[
\int_J f^2 \gamma = \int_J f^2 + \kappa^{-N} \int_J f^2 |\nabla r|^2
\]

\[
\leq \int_J f^2 + c_2 \kappa^{-N} s^{2\beta} \int_J |\nabla f|^2 + c_3 s^{2\beta-d_w} \int_J f^2
\]

\[
\leq c_4 \kappa^{-N} s^{2\beta} \int_J |\nabla f|^2 + c_5 s^{2\beta-d_w} \int_J f^2.
\]

The result now follows from substituting this in the last term of (3.10). \qed

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We now fix $R \in (2, 2 + 2\beta/d)$.

**Theorem 3.9.** Let $Q \subset \mathcal{P}_N$ be the union of a finite number of disjoint special $\mathcal{P}_N$ cubes each of side length $s \leq l_x^2$. Let $f$ and its gradient be square integrable over $Q$. Then

$$
\left( \kappa^{-N} \int_Q |f|^R \right)^{1/R} \leq c_1(R) \left[ \zeta^{-N} \kappa^{-N} \int_Q |\nabla f|^2 + \kappa^{-N} s^{-d_w} \int_Q f^2 \right]^{1/2}.
$$

(3.16)

**Proof.** If $J_i$ are the cubes with $\cup J_i = Q$ then, applying Corollary 3.8 to each of the $J_i$,

$$
\kappa^{-N} \int_Q |f|^R = \sum_{i=1}^M \kappa^{-N} \int_{J_i} |f|^R \leq c_2 \sum_{i=1}^M \left( \int_{J_i} |\nabla f|^2 + s^{-d_w} \kappa^{-N} \int_{J_i} f^2 \right)^{R/2}.
$$

If $p > 1$ and $x_i > 0$ then $\sum x_i^p \leq \left( \sum x_i \right)^p$. So

$$
\kappa^{-N} \int_Q |f|^R \leq \left( \sum_{i=1}^M \left[ \zeta^{-N} \kappa^{-N} \int_{J_i} |\nabla f|^2 + s^{-d_w} \kappa^{-N} \int_{J_i} f^2 \right] \right)^{R/2} = \left( \zeta^{-N} \kappa^{-N} \int_Q |\nabla f|^2 + s^{-d_w} \kappa^{-N} \int_Q f^2 \right)^{R/2}.
$$

\[ \square \]

4. **Harnack inequality.**

In this section we prove Theorem 1.1. We look at the operator $\mathcal{L} = \sum_{i,j=1}^d \partial_i (a_{ij}(x) \partial_j)$, where the $a_{ij}$ are bounded, strictly elliptic, and smooth. On the boundaries of $\mathcal{P}_N$ we impose conormal reflection. We will show the Harnack inequality for nonnegative $\mathcal{L}$-harmonic functions with bounds that do not depend on the smoothness of the $a_{ij}$.

The following result is proved exactly as in Moser [M1], Lemma 4.

**Proposition 4.1.** Let $D \subset \mathcal{P}_N$ be a domain in $\mathcal{P}_N$, and suppose $u$ is $\mathcal{L}$-harmonic in $D$, $v = u^k$, where $k \in \mathbb{R}$, $k \neq 1/2$, and $\eta$ is supported in $D^o$. Suppose the gradient of $\eta$ is square integrable over $D$. Then

$$
\int_D \eta^2 |\nabla v|^2 dx \leq c_1 \left( \frac{2k}{2k-1} \right)^2 \int_D |\nabla \eta|^2 v^2 dx.
$$

Now let $u$ be $\mathcal{L}$-harmonic and non-negative in $\mathcal{P}_N \cap [0, l_x]^d$; we assume $u \neq 0$. The usual Harnack inequality in $\mathbb{R}^d$, combined with the connectedness hypothesis (H2) implies that $u$
and $u^{-1}$ are continuous and bounded (by a constant $c(u, N, \varepsilon)$ depending on $u$, $N$ and $\varepsilon$) on $\mathcal{P}_N \cap [0, l_\mathcal{F} - \varepsilon]^d$ for every $\varepsilon > 0$. By Proposition 4.1, the gradient of powers of $u$ will be square integrable over bounded subsets of $\mathcal{P}_N$.

Let $y_0 = 1$ and $y_k = 1 - \sum_{i=1}^k l_\mathcal{F}^i$ for $1 \leq k \leq \infty$, and set

$$Q_k = [0, y_k]^d \cap \mathcal{P}_N, \quad 0 \leq k \leq \infty.$$ 

Since $l_\mathcal{F} \geq 3$, then $y_\infty \geq \frac{1}{2}$. Note that each $Q_k$ is a $\mathcal{P}_N$-cube and is the union of at most $l_\mathcal{F}^{kd_l}$ special $\mathcal{P}_N$ cubes each of side length $l_\mathcal{F}^{-k}$. Note also that $\text{dist}(Q_{k+1}, \mathcal{P}_N - Q_k) = l_\mathcal{F}^{-(k+1)}$.

**Proposition 4.2.** Let $v$ be either $u$ or $u^{-1}$. There exists $c_1$ such that if $0 < q < 2$, then

$$\sup_{Q_\infty} v^{2q} \leq c_1 \kappa^{-N} \int_{Q_0} (\zeta^{-N} |\nabla v|^2 + v^{2q}).$$

**Proof.** For $0 \leq k < \infty$ let

$$r_k = U(\cdot, Q_{k+1}, Q_k), \quad \gamma_k = 1 + \zeta^{-N} |\nabla r_k|^2.$$ 

Let $f = u^p$, where $p \in \mathbb{R}$, $p \neq \frac{1}{2}$. Applying Theorem 3.9 we have, writing $S = R/2$,

$$\left(\kappa^{-N} \int_{Q_{k+1}} f^{2S} \gamma_{k+1}\right)^{1/S} \leq c_2 \kappa^{-N} \int_{Q_{k+1}} \zeta^{-N} |\nabla f|^2 + l_\mathcal{F}^{(k+1)d_w} \kappa^{-N} \int_{Q_{k+1}} f^2. \quad (4.1)$$

We start with the first term on the right-hand side of (4.1). If $x \in Q_{k+1}$ then there exists a special $\mathcal{P}_N$ cube $I$ of side $l_\mathcal{F}^{-(k+1)}$ such that $x \in I \subset Q_{k+1}$. Then $I^* \subset Q_k$, so by Lemma 2.2(c) we have $r_k \geq c_3 l_\mathcal{F}^{-kd_w}$ on $Q_{k+1}$. Hence

$$\kappa^{-N} \int_{Q_{k+1}} \zeta^{-N} |\nabla f|^2 \leq c_4 \kappa^{-N} \zeta^{-N} l_\mathcal{F}^{2kd_w} \int_{Q_{k+1}} |\nabla f|^2 r_k^2$$

$$\leq c_4 \kappa^{-N} \zeta^{-N} l_\mathcal{F}^{2kd_w} \int_{Q_k} |\nabla f|^2 r_k^2$$

$$\leq c_5 \left(\frac{2p}{2p-1}\right)^2 \kappa^{-N} \zeta^{-N} l_\mathcal{F}^{2kd_w} \int_{Q_k} f^2 |\nabla r_k|^2$$

$$\leq c_5 \left(\frac{2p}{2p-1} + 1\right)^2 \kappa^{-N} l_\mathcal{F}^{2kd_w} \int_{Q_k} f^2 \gamma_k. \quad (4.2)$$

Here we used Proposition 4.1 in the third line. If $c_5$ is taken large enough, the right hand term in (4.2) also dominates the final term in (4.1). Therefore,

$$\left(\kappa^{-N} \int_{Q_{k+1}} f^{2S} \gamma_{k+1}\right)^{1/S} \leq c_6 \left(\frac{2p}{2p-1} + 1\right)^2 l_\mathcal{F}^{2kd_w} \left(\kappa^{-N} \int_{Q_k} f^2 \gamma_k\right). \quad (4.3)$$

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Choose $q' > 0$ such that $\inf_{m \in \mathbb{Z}} |q'S^m - \frac{1}{2}| \geq c_7 > 0$. Suppose first that $q_0 = q'S^{-i}$ for some $i$. Let $p_n = 2q_0S^n$ for $n \geq 0$, and write

$$\Psi_k = \left[\kappa^{-N} \int_{Q_k} v^{p_k \gamma_k}\right]^{1/p_k}.$$

Note that $p_{k+1}/2S = p_k/2$. Applying (4.3) to $f = v^{p_{k+1}/(2S)} = v^{p_k/2}$ we have

$$\Psi_{k+1}^{p_{k+1}/S} = \left(\kappa^{-N} \int_{Q_{k+1}} v^{p_{k+1} \gamma_{k+1}}\right)^{1/S} \leq c_8 l^{2kd_w} \left(\kappa^{-N} \int_{Q_k} v^{p_k \gamma_k}\right) = c_8 l^{2kd_w} \Psi_k,$$

or

$$\Psi_{k+1} \leq \left(c_8 l^{2kd_w}\right)^{1/p_k} \Psi_k.$$

Hence for every $m$

$$\log \Psi_m \leq \log \Psi_0 + \sum_{k=1}^{m} p_k^{-1} \log(c_8 l^{kd_w}). \quad (4.4)$$

As the sum in (4.4) converges, and $\sup_{Q_\infty} v \leq \limsup_{m \to \infty} \Psi_m$, we have

$$\sup_{Q_\infty} v \leq c_9 \left(\kappa^{-N} \int_{Q_0} v^{2q_0 \gamma_0}\right)^{1/(2q_0)}. \quad (4.5)$$

Now let $q \in (0, 2)$. We can take $q_0 = q'S^{-i} < q$. Then by Hölder’s inequality, and Lemma 3.1

$$\int_{Q_0} v^{2q_0 \gamma_0} \leq \left(\int_{Q_0} v^{2q \gamma_0}\right)^{q_0/q} \left(\int_{Q_0} \gamma_0\right)^{1-q_0/q} \leq c_{10} \left(\int_{Q_0} v^{2q \gamma_0}\right)^{q_0/q}.$$

Thus

$$\sup_{Q_\infty} v^{2q} \leq c_{11} \int_{Q_0} v^{2q \gamma_0}.$$

By Corollary 3.5 this implies

$$\sup_{Q_\infty} v^{2q} \leq c_{12} \kappa^{-N} \int_{Q_0} (\zeta^{-N} |\nabla v|^2 + v^{2q}).$$

$\square$

In the argument above we were tied to the cubes $Q_k$ since we needed to use Theorem 3.9. However, in the remainder of this section it will be more convenient to use the balls $B(x, r)$. An easy covering argument gives us

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Corollary 4.3. Let $u > 0$ be $\mathcal{L}$-harmonic in $B(x_0, 3)$. There exists $c_1$, independent of $u$ and $x_0$, such that for $0 < q < 2$ and $v = u$ or $v = u^{-1}$

$$
\sup_{B(x_0, 1)} v^{2q} \leq c_1 \zeta^{-N} \int_{B(x_0, 2)} (\zeta^{-N} |\nabla v^q|^2 + v^{2q}). \quad (4.6)
$$

We now follow the ideas of Moser [M2] to link the $L^\infty$ norms of $u$ and $u^{-1}$. Fix $x_0 \in \mathcal{P}_N$, and write $B(r) = B(x_0, r)$. Let $u > 0$ be $\mathcal{L}$-harmonic in $B(x_0, 4)$. $v$ is either $u$ or $u^{-1}$.

Corollary 4.4. Let $1/2 \leq s < t \leq 4$ and let $r_{st} = U(\cdot, B(s), B(t)), \gamma_{st} = 1 + \zeta^{-N} |\nabla r_{st}|^2$. Then if $0 < q < \frac{1}{3}$,

$$
\sup_{B(s)} v^{2q} \leq c_1 (t - s)^{-d_w - d_f} \zeta^{-N} \int_{B(t)} v^{2q} \gamma_{st}.
$$

Proof. Let $\theta = \frac{1}{4}(t - s), s' = s + 2\theta$. By Corollary 4.3 and scaling, if $B(x, 3\theta) \subset B(4)$ then

$$
\sup_{B(x, \theta)} v^{2q} \leq c_2 \zeta^{-N} \theta^{-d_w - d_f} \int_{B(x, 2\theta)} |\nabla v^q|^2 + c_2 \theta^{-d_f} \zeta^{-N} \int_{B(x, 2\theta)} v^{2q}. \quad (4.7)
$$

We can cover $B(s)$ by a collection of balls $B(x_i, 2\theta) \subset B(s')$ such that no point in $B(s')$ is contained in more than $c_3$ of these balls. So by (4.7)

$$
\sup_{B(s)} v^{2q} \leq c_4 \zeta^{-N} \theta^{-d_w} \int_{B(s')} |\nabla v^q|^2 + c_2 \theta^{-d_f} \zeta^{-N} \int_{B(s')} v^{2q}. \quad (4.8)
$$

By Lemma 2.2(d) $r_{st} \geq c_4 \theta^{d_w}$ on $B(s')$. Since $r_{st}$ is supported on $B(t)^o$, we have by Proposition 4.1,

$$
\zeta^{-N} \int_{B(s')} |\nabla v^q|^2 \leq c_5 \zeta^{-N} \theta^{-2d_w} \int_{B(s')} |\nabla v^q|^2 r_{st}^2 \\
\leq c_6 \zeta^{-N} \theta^{-2d_w} \int_{B(t)} |\nabla v^q|^2 r_{st}^2 \\
\leq c_7 \theta^{-2d_w} \int_{B(t)} |\nabla r_{st}|^2 v^{2q} \\
\leq c_7 \theta^{-2d_w} \int_{B(t)} v^{2q} \gamma_{st}.
$$

Combining this with (4.8) and noting that $\theta \leq 1$ completes the proof. \hfill \Box

Now let $w = \log u$. We will need the following estimate.
Proposition 4.5. Suppose \( h \in [1/2, 2] \) and \( w = \log u \). There exists \( c_1 \) such that

\[
\zeta^{-N} \int_{B(h)} |\nabla w|^2 \leq c_1 |B(h)|.
\]

Proof. Again, this is essentially Moser’s proof. Let \( \phi = U(\cdot, B(h), B(2h)) \), and note that by Lemma 2.2 \( \phi \geq c_2 \) on \( B(h) \). So

\[
\int_{B(h)} |\nabla w|^2 \leq c_3 \int \phi^2 |\nabla w|^2.
\]

We write

\[
0 = \int \frac{\phi^2}{u} \mathcal{L} u = -\int \nabla (\phi^2 / u) \cdot a \nabla u
\]

\[
= -\int \left( 2 \frac{\phi}{u} \nabla \phi \cdot a \nabla u - \frac{\phi^2}{u^2} \nabla u \cdot a \nabla u \right)
\]

\[
= -2 \int \phi \nabla \phi \cdot a \nabla w + \int \phi^2 \nabla w \cdot a \nabla w.
\]

So

\[
\int \phi^2 |\nabla w|^2 \leq c_4 \int |\nabla \phi \cdot a \nabla w| \leq c_5 \left( \int |\nabla \phi|^2 \right)^{1/2} \left( \int \phi^2 |\nabla w|^2 \right)^{1/2}.
\]

Dividing and squaring,

\[
\int \phi^2 |\nabla w|^2 \leq c_6 \int |\nabla \phi|^2,
\]

and by Lemma 3.1, \( \zeta^{-N} \int |\nabla \phi|^2 \leq c_7 |B(h)| \). \( \square \)

For \( \frac{1}{2} \leq h \leq 4 \), let \( \alpha(h) = \frac{1}{|B(h)|} \int_{B(h)} w \).

Corollary 4.6. Let \( \frac{1}{2} \leq s < t \leq 1 \). Then

\[
\int_{\{|w - \alpha(2)| > A\} \cap B(s)} \gamma_{st} \leq \frac{c_1 \zeta^{-N}}{A^2}.
\]

Proof. Note first that

\[
\int_{\{|w - \alpha(2)| > A\} \cap B(s)} \gamma_{st} \leq \int_{\{|w - \alpha(2)| > A\} \cap B(1)} \gamma_{st}.
\]

By Chebyshev’s inequality,

\[
\int_{\{|w - \alpha(2)| > A\} \cap B(1)} \gamma_{st} \leq A^{-2} \int_{\{|w - \alpha(2)| > A\} \cap B(1)} |w - \alpha(2)|^2 \gamma_{st}
\]

\[
\leq A^{-2} \int_{B(1)} |w - \alpha(2)|^2 \gamma_{st}.
\]

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Now apply Corollary 3.4(a) with $Q(\ell) = B(s)$, $Q(k) = B(t)$, $I = B(1)$ and $I^* = B(2)$; we have
\[
\int_{B(1)} (w - \alpha(2))^2 \gamma_{st} \leq c_2 \zeta^{-N} \int_{B(2)} |\nabla w|^2,
\]
and by Proposition 4.5 this is bounded by $c_3 \kappa^{-N}$. □

Without loss of generality, let us multiply $u$ by a constant so that $\int_{B(2)} \log v = \alpha(2) = 0$. Recall that $v$ is either $u$ or $u^{-1}$ and define
\[
\varphi(h) = \sup_{B(h)} \log v.
\]

**Lemma 4.7.** If $\frac{1}{2} \leq s < t \leq 1$, then
\[
\varphi(s) \leq \frac{3}{4} \varphi(t) + c_1 (t - s)^{-d_f - d_w}.
\] (4.9)

*Proof.* Fix $t$ and write $\varphi$ for $\varphi(t)$. Let $c_2 > e$ satisfy $c_2 = 6 \log c_2$. If $\varphi(t) \leq c_2$ then as $\varphi(\cdot)$ is increasing
\[
\varphi(s) \leq \varphi(t) \leq \frac{3}{4} \varphi(t) + \frac{1}{4} c_2,
\]
so that (4.9) holds provided $c_1 \geq c_2/4$.

Now suppose $\varphi > c_2$. By Corollary 4.6 and the fact that $v^p \leq e^{p \varphi}$ on $B(t)$,
\[
\int_{B(t)} v^{2p} \gamma_{st} = \int_{B(t) \cap \{ \log v \geq \varphi/2 \}} v^{2p} \gamma_{st} + \int_{B(t) \cap \{ \log v < \varphi/2 \}} v^{2p} \gamma_{st}
\]
\[
\leq e^{2p \varphi} \int_{B(t) \cap \{ \log v \geq \varphi/2 \}} \gamma_{st} + e^{p \varphi} \int_{B(t) \cap \{ \log v < \varphi/2 \}} \gamma_{st}
\]
\[
\leq \frac{4c_2c_3 e^{2p \varphi}}{\varphi^2} \kappa^{-N} + e^{p \varphi} \int_{B(t)} \gamma_{st}
\]
\[
\leq c_4 \left( \frac{e^{2p \varphi}}{\varphi^2} + e^{p \varphi} \right) \kappa^N.
\]

Let $p = \frac{2}{\varphi} \log \varphi$, so that $e^{p \varphi} = \varphi^2$. As $\varphi > c_2$ we have $p < (2/c_2) \log c_2 = \frac{1}{2}$. So
\[
\int_{B(t)} v^{2p} \gamma_{st} \leq 2c_4 e^{p \varphi} \kappa^N.
\]

Therefore by Corollary 4.4,
\[
\varphi(s) = \frac{1}{2p} \log \left[ \sup_{B(s)} v^{2p} \right]
\]
\[
\leq \frac{1}{2p} \log \left[ c_5 (t - s)^{-d_f - d_w} \kappa^{-N} \int_{B(t)} v^{2p} \gamma_{st} \right]
\]
\[
\leq \frac{1}{2p} \log \left[ c_6 (t - s)^{-d_f - d_w} e^{p \varphi} \right].
\]

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So
\[
\varphi(s) \leq \frac{1}{2} \varphi(t) \left[ 1 + \frac{1}{2} \log(c_6(t-s)^{-d_f-d_w}) \right].
\] (4.10)

Without loss of generality we may take \(c_6\) larger than \(c_2\). If \(\varphi(t) \geq c_6(t-s)^{-d_f-d_w}\), then by (4.10) \(\varphi(s) \leq \frac{3}{4} \varphi(t)\), and (4.9) is satisfied. If, on the other hand, \(\varphi(t) \leq c_6(t-s)^{-d_f-d_w}\), then since \(\varphi(s) \leq \varphi(t)\), we have (4.9) satisfied with \(c_1 = c_6\). \(\square\)

We can now prove the Harnack inequality.

**Theorem 4.8.** There exists \(c_1\) such that if \(u\) is nonnegative and \(L\)-harmonic in \(B(3)\), then
\[
\frac{\sup_{B(1/2)} u}{\inf_{B(1/2)} u} \leq c_1.
\]

**Proof.** We know that \(u\) is continuous and bounded in \(B(2)\); we need to show we can bound the ratio of the supremum of \(u\) to the infimum of \(u\) in \(B(1/2)\) by a constant not depending on \(u\). Multiplying \(u\) by a constant we can assume \(\int_{B(2)} \log u = \alpha(2) = 0\). First let \(v = u\).

Choose \(t_j = 1 - (1/(j + 2))\), so that \(t_0 = 1/2\) and \(t_i \uparrow 1\). Then by Lemma 4.7, writing \(\theta = d_f + d_w\),
\[
\varphi(t_0) \leq \frac{3}{4} \varphi(t_1) + c_2(t_1 - t_0)^{-\theta} \\
\leq (\frac{3}{4})^2 \varphi(t_2) + c_2(t_1 - t_0)^{-\theta} + \frac{3}{4} c_2(t_2 - t_1)^{-\theta} \\
\leq \cdots \\
\leq (\frac{3}{4})^n \varphi(t_n) + \frac{4}{3} \sum_{i=1}^{n} (\frac{3}{4})^i c_2(t_i - t_{i-1})^{-\theta},
\]
for any \(n \geq 0\). Since \(\varphi(t_n) \leq \varphi(1) < \infty\), and
\[
\sum_{i=1}^{\infty} \left(\frac{3}{4}\right)^i c_2(t_i - t_{i-1})^{-\theta} = c_3 < \infty,
\]
we obtain
\[
\sup_{B(1/2)} \log u \leq c_3.
\]

Now let \(v = u^{-1}; \log v = -\log u\) so we still have \(\int_{B(2)} \log v = 0\). The same argument as above now implies \(\sup_{B(1/2)} \log v \leq c_3\), or
\[
\inf_{B(1/2)} \log u \geq -c_3.
\]
Combining we deduce
\[
e^{-c_3} \leq \inf_{B(1/2)} u \leq \sup_{B(1/2)} u \leq e^{c_3},
\]
which is what we wanted to prove. \(\square\)

Theorem 1.1 follows from Theorem 4.8 by a scaling and covering argument.

Standard arguments now yield

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Corollary 4.9. Suppose $D \subset E$ are open subsets of $\mathcal{P}$. There exists $c_1$ depending only on the ratio of dist $(\partial_a D, \partial_a E)$ to the diameter of $D$ such that if $u$ is nonnegative and harmonic in $E$, then
\[ u(x) \leq c_1 u(y), \quad x, y \in D. \]

5. Heat kernel estimates.

In this section we study the fundamental solutions $q(t, x, y)$ of the heat equation in $\mathcal{P}$:
\[ \frac{\partial u}{\partial t}(x, t) = \mathcal{L}u(x, t), \]
where $u$ has conormal reflection on $\partial \mathcal{P}$. In probabilistic terms $q(t, x, y)$ is the transition density of the diffusion $X$ on $\mathcal{P}$ with generator $\mathcal{L}$. We continue to assume the $a_{ij}$ are smooth, although our bounds will not depend on the smoothness.

Let $D$ be a domain in $\mathcal{P}$. Let $u_{t, D}(x, y)$ be the Green function for $\mathcal{L}$ for the process killed on exiting $D$ and let $u_{\Delta, D}$ be the corresponding Green function for reflecting Brownian motion in $D$ killed on hitting $\partial_a D$. Let $C_{\mathcal{L}, D}(A)$ and $C_{\Delta, D}(A)$ be the capacities of a set $A \subset D$ with respect to $\mathcal{L}$ and $\Delta$ respectively. We have
\[ C_{\mathcal{L}, D}(A) = \inf \left\{ \int_D \nabla f \cdot a \nabla f : f = 0 \text{ on } \partial \mathcal{P} - D, f = 1 \text{ on } A \right\} \]
\[ C_{\Delta, D}(A) = \inf \left\{ \int_D | \nabla f |^2 : f = 0 \text{ on } \partial \mathcal{P} - D, f = 1 \text{ on } A \right\}. \]

It is immediate from these definitions that
\[ \lambda_1 C_{\Delta, D}(A) \leq C_{\mathcal{L}, D}(A) \leq \lambda_2 C_{\Delta, D}(A), \quad (5.1) \]
where $\lambda_1, \lambda_2$ depend only on the bounds on the matrix $a$ in (1.1).

Let $Y$ be either of the processes $W$ or $X$. We write $u_D, C_D$ for the Green’s functions and capacities for $Y$, and $\tau_D$ for the exit time of $Y$ from $D$. We define $U_D \mu(x) = \int u_D (x, y) \mu(dy)$.

Lemma 5.1. Let $x \in \mathcal{P}, R > 0, D = B(x, R)$, and $B_n = B(x, 2^{-n} R)$ for $n \geq 0$. Then
\[ c_1 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1} \leq \mathbb{E}^x \tau_D \leq c_2 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1}. \quad (5.2) \]

Proof. Note first that $u_D(x, \cdot)$ is continuous and harmonic on $D - \{x\}$, and is zero on $\{y : d(x, y) = R\} = \partial_a D$. So by the maximum principle it follows that $u_D(x, \cdot)$ attains its maximum on $B_{n-1} - B_n$ at a point $z_n$ with $d(x, z_n) = 2^{-n} R$.  


Let $\mu_n$ be the capacitary measure for $\overline{B}_n$. Thus $C_D(B_n) = \mu_n(\overline{B}_n)$ and $U_D \mu_n \leq 1$ on $D$, and equals 1 on $B_n$. We know that $\mu_n$ is concentrated on $\partial \overline{B}_n$. We have

$$1 = U_D \mu_n(x) = \int_{\overline{B}_n} u_D(x, z) \mu_n(dz) \leq u_D(x, z_n) C_D(B_n). \quad (5.3)$$

Now $u_D(z_n, \cdot)$ is harmonic in $B_{n+1}$, so by the Harnack inequality Corollary 4.9 we have $u_D(z_n, y) \asymp u_D(z_n, x)$ on $B_{n+1}$. Therefore

$$1 \geq U_D \mu_{n+1}(z_n) = \int_{\overline{B}_n} u_D(y, z_n) \mu_{n+1}(dy) \geq c_3 u_D(z_n, x) C_D(B_{n+1}). \quad (5.4)$$

Now write $A_n = B(z_n, 2^{-(n+1)}R)$. Note that $A_n \subset B_n - B_{n+1}$, so that the sets $A_n$ are disjoint. The estimate (2.2) implies that $A_n$, $B_n$ and $B_{n-1}$ all have comparable volume. By Corollary 4.9 we have $u_D(x, y) \asymp u_D(x, z_n)$ on $A_n$.

Using these estimates we have

$$\mathbb{E}^x \tau_D = \int_D u_D(x, y) \geq \sum_{n=1}^{\infty} \int_{A_n} u_D(x, y) dy \geq c_4 \sum_{n=1}^{\infty} \int_{A_n} u_D(x, z_n) dy \geq c_5 \sum_{n=1}^{\infty} |B_n| C_D(B_n)^{-1}. \quad (5.5)$$

Also,

$$\mathbb{E}^x \tau_D = \sum_{n=1}^{\infty} \int_{B_{n-1} - B_n} u_D(x, y) dy \leq c_6 \sum_{n=1}^{\infty} \int_{B_{n-1} - B_n} u_D(x, z_n) dy \leq c_6 \sum_{n=1}^{\infty} |B_{n-1}| u_D(x, z_n) \leq c_7 \sum_{n=1}^{\infty} |B_{n+1}| C_D(B_{n+1})^{-1} \leq c_7 \sum_{m=1}^{\infty} |B_m| C_D(B_m)^{-1} \quad (5.6)$$

Let $\tau_D^L$ be the time for the process $X_t$ associated to $\mathcal{L}$ to exit $D$, and let $\tau_D^A$ be the analogous time for the Brownian motion $W_t$ on $\mathcal{P}$ to exit $D$. Recall from Section 2 the definition $\psi(r) = r^d w \vee \zeta N r^2$, and from Lemma 2.2 that if $D = B(x_0, R)$ then

$$\mathbb{E}^x \tau_D^L \leq c_1 \psi(R), \quad x \in D^*, \quad \mathbb{E}^x \tau_D^A \geq c_2 \psi(R), \quad x \in D.$$

We have the same bounds for the exit times of $X$. 26
Corollary 5.2. There exist $c_1, c_2$ such that
\[
\mathbb{E}^x \tau_{\mathcal{D}^* x}^c \leq c_1 \psi(R), \quad x \in \mathcal{D}^*, \\
\mathbb{E}^x \tau_{\mathcal{D}^* x}^c \geq c_2 \psi(R), \quad x \in \mathcal{D}.
\]

Proof. This is immediate from Lemma 5.1 and the comparison for capacities given by (5.1).

\[\square\]

Theorem 5.3. Let $q(t, x, y)$ denote the transition densities for $X_t$. There exist $c_1, \ldots, c_8 \in (0, \infty)$ such that if $x, y \in \mathcal{P}$ and

(a) $t \geq \max(1, |x - y|)$, then
\[
c_1 t^{-d_*/2} \exp \left( -c_2 \left( \frac{|x - y|^{d_w}}{t} \right)^{1/(d_w - 1)} \right) \\
\leq q(t, x, y) \leq c_3 t^{-d_*/2} \exp \left( -c_4 \left( \frac{|x - y|^{d_w}}{t} \right)^{1/(d_w - 1)} \right);
\]

(b) if $t \leq 1$, then
\[
c_5 t^{-d/2} \exp (-c_6 |x - y|^2/t) \leq q(t, x, y) \leq c_7 t^{-d/2} \exp (-c_8 |x - y|^2/t).
\]

(c) if $t \geq 1, |x - y| > t$, then
\[
c_5 t^{-d_*/2} \exp (-c_6 |x - y|^2/t) \leq q(t, x, y) \leq c_7 t^{-d_* /2} \exp (-c_8 |x - y|^2/t).
\]

Proof. The upper bound follows the proof in [BB3], Section 6, exactly, using Corollary 5.2. The lower bound is done as follows:

Just as in [BB3], (6.23), we have $q(t, x, x) \geq c_0 t^{-d_*/2}$ for $t \geq 1$. Let $D$ be a $\mathcal{P}$-cube of side length $t^{d_w}$ and let $q_D(t, x, y)$ denote the transition densities for $X_t$ killed on exiting $D$. $X$ is a symmetric process, so there is an eigenvalue expansion for $q_D$:
\[
q_D(t, x, y) = \sum_{i=1}^{\infty} e^{-\lambda_i t} \varphi_i(x) \varphi_i(y).
\]

As in [BB2], Section 7, we deduce (6.21) and (6.22) of [BB3]. Since $q(t, x, y) \geq q_D(t, x, y)$, we argue as in [BB3], Theorem 6.9, and derive the lower bound.

We also have a parabolic Harnack inequality for $\mathcal{P}$. The statement and proof are the same as in [BB3], Theorem 7.12.

Remark. Using the results of this section we can construct a diffusion process on $\mathcal{F}$ corresponding to $\mathcal{L}$. As in [BB1], Section 5, we use Corollary 5.2 to obtain a tightness estimate. The Harnack inequality, following [BB1], Section 5, implies that $\lambda$-resolvents are Hölder continuous. We first take a subsequence as in [BB1], Section 6 to construct a process corresponding to $\mathcal{L}$ on $\mathcal{F}$ when the $a_{i,j}$ are smooth. In the case when the $a_{i,j}$ are not smooth, we take $a_{i,j}^n$ smooth satisfying the same bound and uniform ellipticity as the $a_{i,j}$ and take a limit. It is then straightforward, as in [BB3] Section 6, to derive heat kernel bounds and a parabolic Harnack inequality for this process.
References


