is not a semimartingale

by

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Let \( B \) be a one-dimensional Brownian motion, with \( B_0 = 0 \), and let \( L(a, t), a \in \mathbb{R}, t \geq 0 \) be a continuous version of its local time. We shall show that the process \( Y \), defined by \( Y_t = L(B_t, t) \), is not a semimartingale. The essence of the proof is the remark that whereas the paths of a continuous semimartingale satisfy a Holder condition of order \( \frac{1}{2} - \epsilon \) almost everywhere, for any \( \epsilon > 0 \), the paths of \( Y \) just fail to satisfy a Holder condition of order \( \frac{1}{2} \).

For a process or function \( X \) set

\[
D^\alpha(X) = \{ t \geq 0 : \limsup_{\epsilon \to 0} \epsilon^{-1/\alpha} |X_{t+\epsilon} - X_t| > 0 \} .
\]

**Lemma** Let \( \alpha > 1 \), and \( f : \mathbb{R}_+ \to \mathbb{R} \) be a function such that \( D^\alpha(f) = \phi \). Let \( \tau(t) \) be an increasing function, and \( g(t) = f(\tau(t)) \). Then \( |D^\alpha(g)| = 0 \).

**Proof** By Lebesgue's density theorem, \( \tau'(t) \) exists and is finite almost everywhere. For such a \( t \)

\[
\limsup_{\epsilon \to 0} \epsilon^{-1/\alpha} |g(t+\epsilon) - g(t)| = \lim_{\delta \to 0} (\tau'(t))^{1/\alpha} \delta^{-1/\alpha} |f(\tau(t) + \delta) - f(\tau(t))| = 0 ,
\]

so that \( t \notin D^\alpha(g) \).
PROPOSITION  Let \( X \) be a continuous semimartingale. Then for \( \alpha > 2 \),
\[
|D^\alpha(X)| = 0 \quad \text{a.s.}
\]

Proof  Let \( X = M + A^+ - A^- \) be the decomposition of \( X \) into the sum of a martingale and the difference of two increasing processes. It is plain that \( D^\alpha(X) \subset D^\alpha(M) \cup D^\alpha(A^+) \cup D^\alpha(A^-) \). By the lemma, setting 
\[
f(t) = t \quad \text{and} \quad \tau(t) = A^+_t \quad \text{or} \quad A^-_t,
\]
we have \( |D^\alpha(A^+)| = |D^\alpha(A^-)| = 0 \).

Now let \( \tau_t \) be the right-continuous inverse of \( <M> \), and 
\[
U_t = \frac{M_{\tau_t}}{\tau_t}.
\]
Then \( U \) is a Brownian motion, and \( M_t = U_{<M>_t} \). By Lévy's Hölder condition on the variation of Brownian paths, for \( \alpha > 2 \),
\[
D^\alpha(U) = \emptyset \quad \text{a.s., and thus, by the lemma,} \quad |D^\alpha(M)| = 0 \quad \text{a.s.}
\]

THEOREM (i)  For each \( t > 0 \), \( B_t \in D^2(L(\cdot,t)) \) a.s.

(ii) \( B^\alpha(Y) \) is of full Lebesgue measure a.s.

(iii) \( Y \) is not a semimartingale.

Proof  From the results of Ray [1] on Brownian local time,
\[
0 \in D^2(L(\cdot,t)) \quad \text{a.s. Let} \quad t \quad \text{be fixed, and} \quad \tilde{B}_s = B_t - B_{t-s} \quad \text{for} \quad 0 \leq s \leq t.
\]
Then \( \tilde{B} \) is a Brownian motion, and if \( \tilde{L} \) denotes its local time, 
\[
\tilde{L}(a,t) = L(B_t - a,t),
\]
so that \( B_t \in D^2(L(\cdot,t)) \) whenever  
\[
0 \in D^2(\tilde{L}(\cdot,t)),
\]
establishing (i).

We may restate (i) as follows: there exist \( \mathcal{B}_t \)-measurable random variables \( A_n \) and \( C \) with \( |A_n - B_t| < 1/n \), and \( C > 0 \) a.s., such that
\[
|L(A_n,t) - L(B_t,t)| \geq |A_n - B_t|^{1/2}C \quad \text{for all} \quad n.
\]

If \( (a_n) \) is a sequence converging to \( 0 \), and
\[
T_n = \inf\{t \geq 0: B_t = a_n\},
\]
then \( P(T_n < a_n^2) = k > 0 \), for some
constant \( k \). Thus \( P(T_n < a_n^2 \text{ for infinitely many } n) = 1 \) by the Borel-Cantelli lemmas, and the Blumenthal 01 law.

Now let \( S_n = \inf \{ u > t: B_u = A_n \} \). By the preceding argument, and the Markov property of \( B \) at \( t \),

\[
S_n - t < (A_n - B_t)^2 \text{ for infinitely many } n, \ a.s.
\]

Thus

\[
\limsup_{n \to \infty} (S_n - t)^{-\frac{1}{2}} |Y_{S_n} - Y_t|
\]

\[
= \limsup_{n \to \infty} (S_n - t)^{-\frac{1}{2}} |L(A_n, t) - L(B_t, t)|
\]

\[
\geq \limsup_{n \to \infty} (S_n - t)^{-\frac{1}{2}} |A_n - B_t|^{\frac{1}{2}} C
\]

\[
\geq C \ a.s.
\]

\[
> 0 \ a.s.
\]

Therefore \( t \in D^2(Y) \ a.s. \), and (ii) follows by a Fubini argument. (iii) is an immediate consequence of (ii) and the proposition.

Reference