On Countable Dense Random Sets

by

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We shall discuss point processes whose realizations consist typically of a countable dense set of points. In particular, we discuss when such a process may be regarded as Poisson.

The most primitive way to describe a point process on \([0,\infty)\) is as a subset \(B\) of \(\Omega \times [0,\infty)\), where the section \(B_\omega\) represents the times of the "points" in realisation \(\omega\). In the locally finite case, there are the more familiar descriptions using the counting process

\[
N_t(\omega) = \#(B_\omega \cap [0,t]) \quad \text{(as in [BJ])}
\]

or using the random measure

\[
\xi(\omega, D) = \#(B_\omega \cap D) \quad \text{(as in [K])}
\]

Our point processes will generally not be locally finite, so we cannot use these familiar descriptions: we revert to describing a process as a subset \(B\).

We first describe an (obvious) construction of a countable dense Poisson process. Let \(\Theta\) be a countable infinite set. Let \((F_t)\) be a filtration (all filtrations are assumed to satisfy the usual conditions). Suppose \(\{S_i^\theta : i \geq 1, \theta \in \Theta\}\) are optional times such that each counting process \(N_t^\theta = \sum_i 1(S_i^\theta \leq t)\) is a Poisson process of rate 1 with respect to \((F_t)\), and suppose the process \(N^\theta\) are independent. Let \(\xi\) be the random measure on \(\Theta \times [0,\infty)\) whose realisation \(\xi(\omega)\) has the set of atoms \(\{(\theta, S_i^\theta(\omega)) : i \geq 1, \theta \in \Theta\}\). Then \(\xi\) describes a uniform Poisson
process on $\Theta \times [0,\infty)$, with respect to $(F_t)$. But we can also think of $\xi$ as a marked point process on the line. That is, each realisation is an a.s. countable dense set 
\[ \{S_{i\theta}(\omega) : i \in I, \theta \in \Theta \} \] of points in $[0,\infty)$, and each point is marked by some $\theta$. The corresponding unmarked process can be described by

1. $B = \{ (\omega, t) : S_{i\theta}(\omega) = t \text{ for some } i, \theta \} = \{ (\omega, t) : \xi(\omega, \Theta \times \{ t \}) = 1 \}$.

Think of $B$ as a $\sigma$-finite Poisson process. We are concerned with the converse procedure: given a set $B$, when can we assign marks $\theta$ to the points of $B$ to construct a uniform Poisson process $\xi$ satisfying (1)? To allow external randomisation is assigning marks, we make the following definitions:

2. **Definition.** $(G_t)$ is an extension of $(F_t)$ if for each $t$
   
   (i) $G_t \subseteq F_t$
   
   (ii) $G_t$ and $F_\infty$ are conditionally independent given $F_t$.

3. **Definition** $B$ is a $\sigma$-finite Poisson process with respect to $(F_t)$ if
   
   (i) $B$ is $(F_t)$-optional
   
   (ii) There exists a uniform Poisson process $\xi$ on $\Theta \times [0,\infty)$ with respect to some extension $(G_t)$ of $(F_t)$ such that (1) holds.
Theorem 4 below gives a more intrinsic description of \( \sigma \)-finite Poisson processes. First we recall some notation. An optional time \( T \) has conditional intensity \( a(\omega,s) \) if \( T \) has compensator \( A_t = \int_0^t a(s)ds \). We may assume \( a(\omega,s) \) is previsible by [D.V. 19]. Replacing \( (\mathcal{F}_t) \) by an extension does not alter the conditional intensity of an \( (\mathcal{F}_t) \)-optional time \( T \).

Recall also the notation

\[
T_D = T \quad \text{on} \quad D
= \infty \quad \text{elsewhere.}
\]

Let \( \lambda \) be Lebesgue measure on \( [0,\infty) \).

(4) THEOREM. Let \( (\mathcal{F}_t) \) be a filtration. Let \( B \) be an optional set whose sections \( B_\omega \) are a.s. countable. The following are equivalent

(a) \( B \) is a \( \sigma \)-finite Poisson process

(b) There exists a family \( (T^n) \) such that

(5) \( T^n \) is optional; the graphs \( [T^n] \) are disjoint;
\[
B = \bigcup [T^n] \quad \text{a.s.;}
\]

(6) \( T^n \) has a conditional intensity, say \( a_n(\omega,s) \);

(7) \( \sum_n a_n(\omega,s) = \infty \quad \text{a.e.} \quad (P \times \lambda) \)

(b') Every family \( (T^n) \) satisfying (5) also satisfies

(6) and (7)
(c) For every previsible set $C$

$$\{\omega : C_\omega \cap B_\omega = \emptyset\} = \{\omega : \lambda(C_\omega) = 0\} \; \text{a.s.}$$

Remark Families satisfying (5) certainly exist, by the section theorem and transfinite induction [D. VI. 33].

The next result comes out of the proof of Theorem 4.

(8) PROPOSITION. Let $\mu$ be a probability measure on $[0, \infty)$ which is equivalent to Lebesgue measure.

(a) Let $(Y_i)$ be i.i.d. with law $\mu$, and let $(F_t)$ be the smallest filtration making each $Y_i$ optional — that is, the filtration generated by the processes $I_{[Y_i, \infty)}$. Then $B = U[Y_i]$ is a $\sigma$-finite Poisson process with respect to $(F_t)$.

(b) Conversely, let $B$ be a $\sigma$-finite Poisson process with respect to some $(F_t)$. Then there exist times $(Y_i)$ such that $B = U[Y_i] \; \text{a.s.}$, $(Y_i)$ are i.i.d. with law $\mu$, and $(Y_i)$ are optional with respect to some extension of $(F_t)$.

Before the proofs, here is an amusing example.

Example There exists a process $X_t$ and filtrations $(F_t)$, $(G_t)$ such that $X$ is optional with respect to each of $(F_t)$ and $(G_t)$, but $X$ is not optional with respect to $F_t \cap G_t$.

To construct the example, let $(Y_i), B, (F_t)$ be as in
part (a) of Proposition 8, and let $X = l_B$. Let $\Pi$ be the set of finite permutations $\pi = (\pi(1), \pi(2), \ldots)$ of $(1, 2, \ldots)$. Since $\Pi$ is countable we can construct a random element $\pi^*$ of $\Pi$ such that $P(\pi^* = \pi) > 0$ for each $\pi \in \Pi$. Take $\pi^*$ independent of $Y = (Y_1, Y_2, \ldots)$. Define $\tilde{Y} = (V_1, V_2, \ldots) = (Y_{\pi^*}(1), Y_{\pi^*}(2), \ldots)$. Let $(F_t)$ be the smallest filtration making each $V_t$ optional. Since $X_t = \Sigma_1(V_1 = t) = \Sigma_1(V_1 = t)$, plainly $X$ is both $(F_t)$- and $(G_t)$-optional. But $F_\infty \cap G_\infty$ is trivial! For let $D \in F_\infty \cap G_\infty$. Then there exist measurable functions $f, g$ such that

$$l_D = f(\tilde{Y}) = g(\tilde{Y}) \text{ a.s.}$$

So $f(\tilde{Y}) = h(\tilde{Y}, \pi^*)$ a.s., where $h(Y_1, Y_2, \ldots; \pi) = g(Y_{\pi}(1), Y_{\pi}(2), \ldots)$

But $\pi^*$ is independent of $\tilde{Y}$ with support $\Pi$, so

$$F(\tilde{Y}) = h(\tilde{Y}, \pi) \text{ a.s., each } \pi \in \Pi.$$

So, putting $G = \{g = 1\}$,

$$D = \{(Y_{\pi}(1), Y_{\pi}(2), \ldots) \in G\} \text{ a.s., each } \pi \in \Pi.$$

Thus $D$ is exchangeable, and so is trivial by the Hewitt-Savage zero-one law.

We now start the proof of Theorem 4. The lemma below shows that (b) and (b') are equivalent.

(9) LEMMA. Let $(T^n)$ be optional times whose graphs $[T^n]$ are disjoint. Let $(\hat{T}^m)$ be a similar family, and suppose $U[T^n] = U[\hat{T}^m]$. Suppose $T^n$ has conditional intensity $a_n$. 
Then \( \hat{a}_m \) has a conditional intensity, \( \hat{a}_m \) say, and
\[ \Sigma \hat{a}_m = \Sigma a_n \text{ a.e. } (P \times \lambda). \]

**Proof.** Put \( U_{m,n} = T^n(I^n = T^m) \). Then \( U_{m,n} \) has a conditional intensity, \( a_{m,n} \) say. It is easy to verify

\[ a_n = \Sigma a_{m,n} \text{ a.e.} \]

\[ \hat{a}_m \equiv \Sigma a_{m,n} \] is the conditional intensity of \( \hat{a}_m \), where the sum is a.e. finite because

\[ \mathbb{E} \left\{ \sum_{n=1}^{N} a_{m,n}(s) ds \right\} = \sum_{n=1}^{N} P(U_{m,n} < \infty) \leq P(T^m < \infty) \leq 1. \]

Hence \( \Sigma a_n = \Sigma \Sigma a_{m,n} = \Sigma \hat{a}_m \equiv \infty \text{ a.e.} \)

Lemmas 10 and 13 show that conditions (b') and (c) are equivalent.

**(10) LEMMA.** For \( B \) as in theorem 4, the following are equivalent

(i) \( \{ \omega : C_\omega \cap B_\omega = \emptyset \} \Rightarrow \{ \omega : \lambda(C_\omega) = 0 \} \) a.s., each previsible \( C \).

(ii) Each family \( (Y^n) \) satisfying (5) also satisfies (6).

**Proof: (ii) implies (i)** Let \( C \) be previsible. Put
\[ T = \inf \{ t : \lambda(C_\omega \cap [0,t]) > 0 \} \]. Then \( T \) is optional, so \( C' = C \cap [0,T] \) is previsible. Now \( \lambda(C'_\omega) = 0 \) a.s. We must prove

**(11)** \( C'_\omega \cap B_\omega = \emptyset \) a.s.
Let \((T^n)\) satisfy (5) and (6). Then

\[
P(T^n \in C_\omega) = E\int_{C_\omega}^{T^n} d\mu_{T^n, \omega} = E\int_{C_\omega}^{T^n} a_n(s) \, ds
\]

\[= 0.
\]

Since \(B = U[T^n]\), (11) follows.

(i) implies (ii). Let \(T\) be optional, \([T] \subset B\). Let \(A_t\) be the compensator of \(T\). From the proof of the Lebesgue decomposition theorem, we can write \(A_t = \hat{A}_t + \int_0^t a(s) \, ds\), where there exists a progressive set \(D\) such that

\[
\lambda(D_\omega) = 0 \quad \text{a.s.; the measure } \, d\hat{\lambda}(\omega) \text{ is carried on } \quad D_\omega \quad \text{a.s.}
\]

Let \(C = \{p(1_D) > 0\}\); then \(C\) is previsible and since

\[
\hat{A}_t = \int_0^t 1_C(s) \, d\hat{\lambda}_s \geq \int_0^t p(1_D)(s) \, d\hat{\lambda}_s = \int_0^t 1_D(s) \, d\hat{\lambda}_s = \hat{A}_t,
\]

and

\[
\int_0^t p(1_D)(s) \, ds = \int_0^t 1_D(s) \, ds = 0,
\]

\(C\) satisfies (12). However

\[
E\hat{\lambda}_\infty = E\int 1_C(s) \, d\hat{\lambda}_s
\]

\[= E\int 1_C(s) \, d\hat{\lambda}_s
\]

\[= P(T \in C_\omega) = 0 \quad \text{by (11)}.
\]
So \( \hat{A} = 0 \).

(13) **LEMMA.** For \( B \) as in Theorem 4, the following are equivalent.

(i) \( \{\omega : C_\omega \cap B_\omega \neq \emptyset\} \supset \{\omega : \lambda(C_\omega) > 0\} \) a.s., each previsible \( C \).

(ii) Each family \((T^n)\) satisfying (5) and (6) also satisfies (7).

**Proof.** (ii) implies (i) Let \( C \) be previsible. Define optional times:

\[
T = \inf\{t : \lambda(C_\omega \cap [0, t]) > 0\},
\]

\[
S = \inf\{t : t \in B_\omega \cap C_\omega\}.
\]

It is sufficient to prove

(14) \( S \leq T \) a.s.

Consider the previsible set \( C' = C \cap (T, S) \)

Let \((T^n)\) satisfy (5), (6). By definition of \( S \), the sets \( \{\omega : T^n \in C'_\omega\} \) are disjoint. So \( \Sigma P(T^n \in C'_\omega) \leq 1 \). But

\[
\Sigma P(T^n \in C'_\omega) = \Sigma E \int_{T^n, \infty} 1_{C'} \, dl_{T^n, \infty}
\]

\[
= \Sigma E \int_{C'} (s) a_n(s) \, ds
\]

\[
= \Sigma E \int_{C'} (s) \Sigma a_n(s) \, ds.
\]

But \( \Sigma a_n = \infty \) a.e., and so \( \lambda(C'_\omega) = 0 \) a.s. But by definition of \( T \), we have \( \lambda(C'_\omega) > 0 \) on \( \{T < S\} \). This proves 14.
(i) implies (ii) Let \( T_n \) satisfy (5), (6). Fix \( N < \infty \).
Consider the previsible set \( H = \{(\omega, s): \Sigma a_n \leq N-1\} \). We must prove \( P \times \lambda(H) = 0 \). Suppose not: then for some \( \varepsilon > 0 \) we have

\[
P(\Omega_0) \geq \varepsilon, \text{ where } \Omega_0 = \{\omega: \lambda(H_\omega) > \varepsilon\}.
\]

Define optional times

\[
S_i = \inf \{t: \lambda(H_\omega \cap [0, t]) > i\varepsilon/N\} \quad i = 0, \ldots, N.
\]

Consider the previsible sets

\[
H^i = H \cap (S_{i-1}, S_i] \quad i = 1, \ldots, N
\]

\[
\bar{H} = H \cap (S_0, S_N].
\]

By construction, \( \lambda(H^i_\omega) = \varepsilon/N \) on \( \Omega_0 \). So by (i),

\[
B_\omega \cap H^i_\omega \text{ is a.s. non-empty on } \Omega_0.
\]

So

\[
E \sum_n 1(T_n \in \bar{H}_\omega) = E \sum_{i=1}^N 1(T_n \in H^i_\omega) \geq N \ P(\Omega_0) \geq N\varepsilon.
\]

But

\[
E \sum_n 1(T_n \in \bar{H}_\omega) = E \sum_n \int_{T_n}^{\infty} dI(\bar{H}_\omega, \omega) = E \int_{T_n}^{\infty} dI(\bar{H}_\omega, \omega)
\]

\[
= E \int_{T_n}^{\infty} l^-(s) \cdot \Sigma a_n(s) \ ds
\]

\[
\leq (N-1) \varepsilon
\]

because \( \Sigma a_n \leq N-1 \) on \( H \), and \( \lambda(\bar{H}_\omega) \leq \varepsilon \) by construction.

This contradiction establishes the result.
It remains to prove that (b) and (a) are equivalent. Recall from [BJ] that optional times \( 0 < S_1 < S_2 < \ldots \) form a Poisson process of rate 1 with respect to \((F_t)\) iff \( S_n \) has conditional intensity \( \mathbf{1}_{(S_{n-1} < s \leq S_n)} \). If moreover this condition holds for each family \( (S_i^\theta)_{i \geq 1}, \theta \in \Theta \), and if the graphs \( \{[S_i^\theta] : i \geq 1, \theta \in \Theta \} \) are disjoint, then the families \( \{[S_i^\theta]_{i \geq 1} : \theta \in \Theta \} \) are independent.

The proof that (a) implies (b) is easy. The family \( (S_i^\theta) \) in (1) plainly satisfy the conditions of (b) with respect to the extension \((G_t^\theta)\). Because (b) implies (b'), we deduce that any \((G_t^\theta)\)-optional family satisfying (5) will also satisfy (6) and (7) with respect to \((G_t^\theta)\). Now, as remarked before, there exists a family satisfying (5) with respect to \((F_t)\); and since conditional intensities are unchanged by extension, this family satisfies (6) and (7) with respect to \((F_t)\).

The proof that (b) implies (a) is harder. There are only two ideas. First, we show how to construct \( S_1 \) with \([S_1] \subset B\) such that \( S_1 \) has exponential law (Lemma 19). Then we can proceed inductively to construct a uniform Poisson process \((S_i^\theta)\). Finally, we must show that \( \bigcup_{\theta \in \Theta} [S_i^\theta] \) exhausts \( B \).

Here is a straightforward technical lemma.

(14) **Lemma.** Let \((Q_i)\) be optional times with conditional intensities \( a_i \). Suppose \( Q_i \to \infty \) a.s. and \([Q_i]\) are disjoint. Let \( T = \min(Q_i) \). Then
\[
T_{(T=Q_i)} \text{ has conditional intensity } a_i \mathbf{1}(s \leq T)
\]
\[
T \text{ has conditional intensity } \sum a_i \mathbf{1}(s \leq T).
\]
Here is an informal description of the external randomisation. Suppose

(15) $T$ is optional, with conditional intensity $a$, $p(\omega,s)$ is previsible, $0 \leq p \leq 1$.

Then we can define $Q$ such that:

if $T = t$ then $Q = t$ with probability $p(\omega,t)$

$= \infty$ otherwise.

It is intuitively obvious that $Q$ has conditional intensity $p.a$. Here is the formal construction and proof.

(16) LEMMA. Let $T, a, p$ be as in (15), on a filtration $(\hat{F}_t)$. Let $U$ be uniform on $[0,1]$, independent of $\hat{F}_\omega$. Define

$$Q = T \text{ if } U \leq p(T) \equiv p(\omega,T(\omega))$$

$$= \infty \text{ otherwise.}$$

Let $G_t$ be the usual augmentation of $G^0_t = \sigma(\hat{F}_t, Q(\omega \leq t))$. Then $(G_t)$ is an extension of $(\hat{F}_t)$, and $Q$ is $(G_t)$-optional with conditional intensity $p.a.$

**Proof** $Q(\omega \leq t) \in \sigma(\hat{F}_t, U)$, and hence $G^0_t = \sigma(\hat{F}_t, U)$, so $(G_t)$ is indeed an extension of $(\hat{F}_t)$. Plainly $Q$ is $(G_t)$-optional. To prove the final assertion, let $S < \infty$ be a $(G_t)$-optional time. It is sufficient to prove

(17) $P(Q < S) = E \int_0^S a(s)p(s)ds$. 
We assert

(18) \( R = S_{(S \leq T)} \) is \((F_t)\)-optional.

For \( \{R < u\} = U \{S < T\} \), and \( \{S < T\} \) is in \( F_t \) since \( T < u \) \( t \) rational

\( G_t \cap \{T > t\} = F_t \cap \{T > t\} \). To prove (17), note that

\( \{Q \leq S\} = \{T \leq S, Q < \infty\} = \{T \leq R, Q < \infty\} = \{T \leq R, T < \infty, U \leq P(T)\} \). So

\[
P(Q \leq S) = P(T \leq R, T < \infty, U \leq P(T))
\]

\[
= E(1_{(T \leq R, T < \infty)} P(U \leq P(T) | F_\infty))
\]

\[
= E 1_{(T \leq R, T < \infty)} P(T) \text{ by the independence of } U
\]

\[
= E \int 1_{(s \leq R)} P(s) \, dl_{[T, \infty)}
\]

\[
= E \int 1_{(s \leq R)} P(s) \, a(s) \, ds.
\]

(17) now follows, as \([S, R] \subseteq [T, \infty)\), and \( a = 0 \) on this set.

(19) LEMMA. Let \((\hat{F}_t)\) be an extension of \((\hat{F}_t)\). Suppose \((T^H)\) satisfies condition (b) with respect to \((\hat{F}_t)\).
Let \( S_0 < \infty \) be \((\hat{F}_t)\)-optional. Then there exists an extension \((G_t)\) of \((\hat{F}_t)\) and a \((G_t)\)-optional time \( S \) with conditional intensity \( 1(S_0 < S \leq S) \) such that \([S] \subseteq U[T^H] \).
Proof Define \( \phi(x) = 1 \quad x \geq 1 \)
\[ = x \quad 0 \leq x < 1 \]
\[ = 0 \quad x \leq 0 \]

Define inductively

\[
P_l(\omega, s) = \phi\left(\frac{1}{a_l(\omega, s)}\right) \, 1(s > s_0) = \frac{1}{\sum_{i=1}^{j-1} a_i \, p_i} \] 

\[ p_j = \phi\left(\frac{1}{a_j}\right) \, 1(s > s_0) \]

Then \( p_j \) is predictable, \( 0 \leq p_j \leq 1 \), and

\[
\sum_{j=1}^{N} a_j p_j = (1 - \sum_{j=1}^{N} a_j) \cdot 1(s > s_0)
\]

By Lemma 16 we can construct extensions \( (G_t^j) \) of \( (f_t) \) and \( (G_t^j) \)-optional times \( Q_j \) such that

\[
[Q_j] < [T^j] \quad ,
\]

\( Q_j \) has conditional intensity \( p_j a_j \).

Then

\[
\sum_{j} P(Q_j < t) = \sum_{j} E \int_{0}^{t} p_j(s) a_j(s) \, ds
\]
\[ = E \int_{0}^{t} a_j(s) p_j(s) \, ds \leq t \text{ by (20)} .
\]
By the Borel-Cantelli lemma, $Q_j \rightarrow \infty$ a.s.

Set $S = \min (Q_j)$, and let $(G_t)$ be the filtration generated by $(G_t^j, j \geq 1)$. By Lemma 14, $S$ has conditional intensity $\sum_j p_j 1_{(s \leq S)}$, and by (20) this equals $1(S_0 < s \leq S)$. For later use, note that, by Lemma 14, $S_{(S\leq T^n)}$ has conditional intensity $p_\nu a_n 1_{(s \leq S)}$. In other words, using (20),

$$T^n_{(T^n = S)} \text{ has conditional intensity}$$

$$\sum_{i=1}^{N} a_i \left[ 1 \wedge \sum_{i=1}^{N-1} a_i \right] 1(S_0 < s \leq S).$$

We can now prove (b) implies (a). Let $(T^1, n)$ satisfy condition (b). By Lemma 19 we can construct extensions $G_t^1, G_t^2, \ldots$ of $F_t$ and $(G_t^1)$-optional times $S^1_i$ such that $[S^1_i] \subset B$ and such that $S^1_i$ has conditional intensity $1(S_{i-1} < s \leq S^1_i)$. Let $F^1$ be the filtration generated by $(G^1: i \geq 1)$. Then $(S^1_i)_{i \geq 1}$ is a Poisson process of rate 1 with respect to $F^1$.

Now let $T^2, n = T^1, n \setminus F^1$ (for any $i$, $T^1, n \neq S^1_i$ for any $i$). We assert that $(T^2, n)$ satisfies (b) with respect to $(F_t)$, for a certain set $B'$. We need only check (7). Write $a_k, n$ for the conditional intensity of $T^k, n$. Write

$$R_{n, i} = T^1, n \setminus S^1_i$$

$$R_n = T^1, n \setminus S^1_i \text{ for some } i.$$
Then

(22) \( R_n \) has conditional intensity \( a_{1,n} - a_{2,n} \geq 0 \).

But \( \mathbb{U}[R_n] = \mathbb{U}[R_n,i] = \mathbb{U}[S_i^1] \), so by Lemma 9

\[
\sum (a_{1,n} - a_{2,n}) = \sum \mathbb{1}_{(S_{i-1}^1 < s \leq S_i^1)} = 1 \text{ a.e.}
\]

Thus condition (7) extends from \((t_1^n, n)\) to \((t_2^n, n)\).

Now we may apply Lemma 19 again to construct an extension \( F^2 \) and \( F^2 \)-optional times \((S_i^2)\) with

\([S_i^2] \subset \mathbb{U}[T_2^n, n] \) and such that \( (S_i^2)_{i \geq 1} \) is again a Poisson process of rate 1.

Continuing, we obtain a uniform Poisson process

\((S_i^k : i,k \geq 1)\) on \(\{1,2,\ldots\} \times [0,\infty)\). By construction

\(\mathbb{U}[S_i^k] \subset B\), but we must show there is a.s. equality. Thus

we must show that, for each \( n \),

(23) \( P(T_k^n < \infty) = \mathbb{E} \int a_{k,n}(s) \, ds + 0 \) as \( k \to \infty \).

Define

\[
R_n^k = T_n^k, n = S_i^k \text{ for some } i.
\]

As at (22), \( R_n^k \) has conditional intensity \( a_{k,n} - a_{k+1,n} \).

But from (21), \( R_n^k \) has conditional intensity \( (1 - \sum_{i=1}^{k-1} \mathbb{E} a_{k,i}) - \sum_{i=k+1}^{n} \mathbb{E} a_{k,i} \).

So

(24) \( \mathbb{E} \int (a_{k,N} - a_{k+1,N}) \, ds = \mathbb{E} \int (1 - \sum_{i=1}^{N} \mathbb{E} a_{k,i}) - (1 - \sum_{i=1}^{N-1} \mathbb{E} a_{k,i}) \, ds \).
Now \( a_{k,m} + a_{\infty,n} \), say, as \( k \to \infty \). Suppose, inductively, that (23) holds for \( n < N \). As \( k \to \infty \) the left side of (24) tends to 0, and the right side tends to \( \mathbb{E}\int (1 \wedge a_{\infty,N}) \, ds \) by the inductive hypothesis. Thus \( a_{\infty,N} = 0 \) a.e, so (23) holds for \( N \).

**Proof of Proposition 8.** Put \( f(t) = \frac{F'(t)}{1-F(t)} \), where \( F \) is the distribution function of \( \mu \).

From [BJ], if \( Y \) has conditional intensity \( f(s)1_{(s \leq Y)} \) then \( Y \) has law \( \mu \): conversely, if \( Y \) has law \( \mu \) then \( Y \) has conditional intensity \( f(s)1_{(s \leq Y)} \) with respect to the smallest filtration making \( Y \) optional. Thus the random variables \( (Y_i) \) in part (a) of Proposition 8 satisfy condition (b) of Theorem 4, so \( U[Y_i] \) is indeed a \( \sigma \)-finite Poisson process.

Part (b) is similar to , but simpler than, the proof that (b) implies (a) in Theorem 4. Let \( B \) be a \( \sigma \)-finite Poisson process, and let \( (T^{1,n}) \) satisfy condition (b) of Theorem 4. Lemma 19 showed how to construct an optional time \( S \) with conditional intensity \( 1_{(s \leq S)} \). Essentially the same argument shows we can construct \( Y_1 \) with conditional intensity \( f(s)1_{(s \leq Y_1)} \), and hence with law \( \mu \). Put \( T^{2,n} = T^{1,n} \cap (T^{1,n} \neq Y_1) \), and continue. We obtain i.i.d. variables \( (Y_k) \), with \( U[Y_k] \subset B \): arguing as at (23), we show that there is a.s. equality.
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