

# Stability of parabolic Harnack inequalities on metric measure spaces

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**Abstract.** Let  $(X, d, \mu)$  be a metric measure space with a local regular Dirichlet form. We give necessary and sufficient conditions for a parabolic Harnack inequality with global space-time scaling exponent  $\beta \geq 2$  to hold. We show that this parabolic Harnack inequality is stable under rough isometries. As a consequence, once such a Harnack inequality is established on a metric measure space, then it holds for any uniformly elliptic operator in divergence form on a manifold naturally defined from the graph approximation of the space.

**Key words and Phrases.** Harnack inequality, volume doubling, Green functions, Poincaré inequality, Sobolev inequality, rough isometry, anomalous diffusion

**Short title.** Harnack inequalities on metric spaces

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**1. Introduction.** Let  $(X, d, \mu)$  be a metric measure space with a local regular Dirichlet form  $(\mathcal{E}, \mathcal{F})$ . (See Section 2 for definitions of the terms used in the introduction.) Assume that the metric is geodesic and for simplicity assume that  $X$  has infinite diameter. For a typical example let  $X$  be a complete non-compact Riemannian manifold with Riemannian metric, Riemannian measure and  $\mathcal{E}(f, f) = \int_X |\nabla f|^2 d\mu$ . In this paper, we give necessary and sufficient conditions for a parabolic Harnack inequality with ‘anomalous’ space-time scaling to hold, and we show the stability of this parabolic Harnack inequality under certain transformations.

For  $\bar{\beta}, \beta \geq 2$ , let  $\Psi(s) = s^{\bar{\beta}}1_{\{s \leq 1\}} + s^{\beta}1_{\{s > 1\}}$ . For  $R > 0$  let  $T = \Psi(R)$ ,  $Q_- = (T, 2T) \times B(x_0, R)$  and  $Q_+ = (3T, 4T) \times B(x_0, R)$ . Let  $\Delta$  be the self-adjoint operator corresponding to  $(\mathcal{E}, \mathcal{F})$ .  $X$  satisfies the *parabolic Harnack inequality*  $\text{PHI}(\Psi)$ , if there exists a constant  $c_1$  such that for any  $x_0 \in X$  and  $R > 0$ , if  $u = u(t, x)$  is a non-negative solution of the heat equation  $\partial_t u = \Delta u$  in  $(0, 4T) \times B(x_0, 2R)$ , then

$$\sup_{Q_-} u \leq c_1 \inf_{Q_+} u.$$

If  $\bar{\beta} = \beta$  we sometimes write  $\text{PHI}(\beta)$  for  $\text{PHI}(\Psi)$ . It is known that such a Harnack inequality is strongly related to detailed estimates of the heat kernel, i.e., the fundamental solution of the heat equation.

When  $\bar{\beta} = \beta = 2$ , the study of the parabolic Harnack inequality has a long history (see [Da, SC2] etc. for details). For any divergence operator  $\mathcal{L} = \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) \frac{\partial}{\partial x_j})$  on  $\mathbb{R}^n$  which is uniformly elliptic, Aronson ([A]) proved that the heat kernel  $p_t(x, y)$  satisfies

$$\frac{c_1}{\mu(B(x, t^{1/2}))} \exp\left(-\frac{d(x, y)^2}{c_1 t}\right) \leq p_t(x, y) \leq \frac{c_2}{\mu(B(x, t^{1/2}))} \exp\left(-\frac{d(x, y)^2}{c_2 t}\right), \quad (1.1)$$

where  $\mu$  is Lebesgue measure (so  $\mu(B(x, t^{1/2}))^{-1} = ct^{-n/2}$ ). It is not hard to derive the parabolic Harnack inequality from (1.1). Similar results hold in the field of global analysis on manifolds. Let  $\Delta$  be the Laplace-Beltrami operator on a complete Riemannian manifold  $M$  with Riemannian metric  $d$  and with Riemannian measure  $\mu$ . Li-Yau ([LiY]) proved that if  $M$  has non-negative Ricci curvature, then the heat kernel  $p_t(x, y)$  satisfies (1.1). A few years later, Grigor’yan ([Gr1]) and Saloff-Coste ([SC1]) refined this result and proved, in conjunction with results by Fabes-Stroock ([FS]) and Kusuoka-Stroock ([KS]), that (1.1) is equivalent to a volume doubling condition (VD) plus a Poincaré inequality (PI(2)). These techniques were then extended to Dirichlet forms on metric spaces in [BM, St1, St2] and to graphs in [Del]. The origin of the ideas and techniques used in this field go back to Nash ([N]) and Moser ([M1, M2, M3]).

Examples where  $\bar{\beta} = \beta > 2$  are given by fractals. The mathematical study of stochastic processes and the corresponding operators on fractals (see, for instance, [B1, Ki]) has shown that on many ‘regular’ fractals, there are naturally defined Dirichlet forms whose heat kernels with respect to the Hausdorff measure  $\mu$  satisfy

$$\begin{aligned} \frac{c_1}{\mu(B(x, t^{1/\beta}))} \exp\left(-\left(\frac{d(x, y)^\beta}{c_1 t}\right)^{\frac{1}{\beta-1}}\right) &\leq p_t(x, y) \\ &\leq \frac{c_2}{\mu(B(x, t^{1/\beta}))} \exp\left(-\left(\frac{d(x, y)^\beta}{c_2 t}\right)^{\frac{1}{\beta-1}}\right), \end{aligned} \quad (1.2)$$

(see [BP, BB1, BB2, FHK, Kum1] etc.). The techniques of [Gr1] and [SC1] do not apply to these spaces. The main obstacle is that the Moser iteration argument used in these papers needs the existence of sufficiently many cut-off functions  $\varphi$  with approximately minimal energy such that the  $L^2$  and  $L^\infty$  norms of  $\nabla\varphi$ , suitably normalized, are comparable. Functions of this kind exist only if  $\bar{\beta} = \beta = 2$ .

Recently, in [BB5], two of the authors established equivalent stable conditions for (1.2) (equivalently, to  $\text{PHI}(\beta)$ ) on graphs. The key idea was to introduce a new inequality  $\text{CS}(\beta)$ , called a cut-off Sobolev inequality, which implies the existence of enough ‘low energy’ cut-off functions on the space. This was quite recently extended to the manifold setting by one of the authors in [B2]. Note that when the process is strongly recurrent, there are simpler equivalent stable conditions (see [BCK, Kum3]), given in terms of electrical resistance.

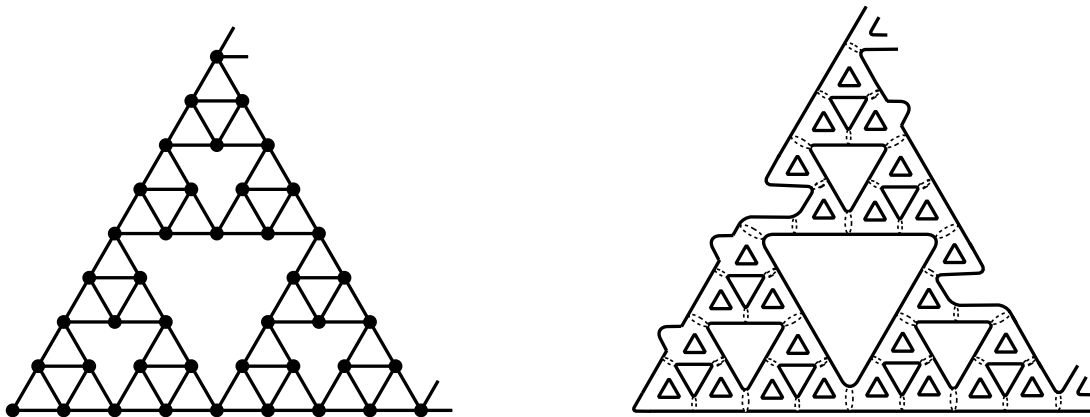


Figure 1. Sierpinski gasket graph and fractal-like manifold

The aim of this paper is twofold. First we extend the results in [BB5, B2] to the general framework of metric measure spaces. One of our main theorems gives necessary and sufficient conditions for  $\text{PHI}(\Psi)$  to hold (Theorem 2.16). The proofs given here are based closely on those in [BB5]. Secondly, we establish the stability of  $\text{PHI}(\Psi)$  under rough isometries and under bounded perturbations, assuming some local regularity on the spaces (Theorem 2.21). Let us look at two examples. The right side of Figure 1 is an example of a fractal-like manifold  $M$ : it is a 2-dimensional Riemannian manifold which is made from the Sierpinski gasket graph (left side of Figure 1) by replacing the edges by tubes of length 1, and by gluing the tubes together smoothly at the vertices. Also, one allows some small bumps, and some of the tubes to be removed. More precisely,  $M$  is a smooth Riemannian manifold which is roughly isometric to the Sierpinski gasket graph. Brownian motion moves on the surface of the tubes. For our second example, Figure 2 is an example of a fractal tiling which is made by the following procedure. First, consider the triangular lattice on  $\mathbb{R}^2$  where each edge is of length 1. We then fill each triangle with a Sierpinski gasket. This fractal tiling is roughly isometric to  $\mathbb{R}^2$ . One can define a Dirichlet form on the space by summing up Dirichlet forms on the gasket. As a consequence of our main theorems and the known heat kernel estimates on the Sierpinski gasket ([BP, Jo]), we have the following: the Dirichlet form for the manifold depicted on the right side of Figure 1 satisfies  $\text{PHI}(\Psi)$  with  $\Psi(s) = s^2 \vee s^{\log 5 / \log 2}$  while the Dirichlet form for the tiling shown in Figure 2 satisfies  $\text{PHI}(\Psi)$  with  $\Psi(s) = s^2 \wedge s^{\log 5 / \log 2}$ .

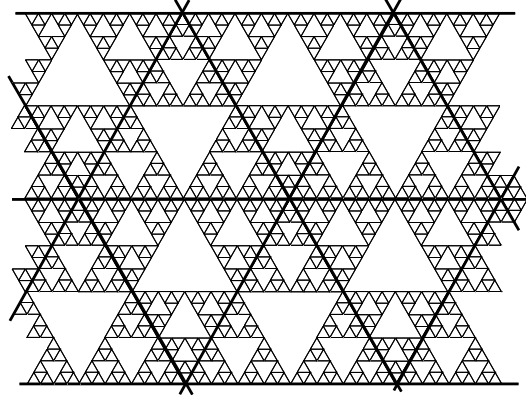


Figure 2. Tiling of Sierpinski gaskets

The organization of the paper is the following. In Section 2, we present the framework and the main theorems. Sections 3 and 4 provide the proof of Theorem 2.16. In Section 5, we give the proof of Theorem 2.21. We also give several applications. Our results, together with Hino's results in [Hin], complete the proof of the singularity of the energy measures of Dirichlet forms on higher dimensional Sierpinski carpets with respect to the Hausdorff measures. Another application gives transition density estimates for the self-similar Brownian motions defined on higher-dimensional Sierpinski carpets by [KZ].

We will use  $c$ , with or without subscripts, to denote strictly positive finite constants whose values are insignificant and may change from line to line.

## 2. Framework and main theorems.

Let  $(X, d)$  be a connected locally compact complete separable metric space. We assume that the metric  $d$  is geodesic: for each  $x, y \in X$  there exists a (not necessarily unique) geodesic path  $\gamma(x, y)$  such that for each  $z \in \gamma(x, y)$ , we have  $d(x, z) + d(z, y) = d(x, y)$ . Let  $\mu$  be a Borel measure on  $X$  such that  $0 < \mu(B) < \infty$  for every ball  $B$  in  $X$ . We write  $B(x, r) = \{y : d(x, y) < r\}$ , and  $V(x, r) = \mu(B(x, r))$ . Note that under the assumptions above, the closure of  $B(x, r)$  is compact for all  $x \in X$  and  $0 < r < \infty$ . For simplicity in what follows, we will also assume that  $X$  has infinite diameter, but similar results (with obvious modifications to the statements and the proofs) hold when the diameter of  $X$  is finite. We will call such a space a *metric measure space*, or a MM space. 'Metric measure space' is not a new term, and we mention [Gr2, GHL, Ha, Kas, Ke] as a sample of recent papers that deal with analysis on such spaces.

Now let  $(\mathcal{E}, \mathcal{F})$  be a regular, strong local Dirichlet form on  $L^2(X, \mu)$ : see [FOT] for details. We denote by  $\Delta$  the corresponding self-adjoint operator; that is, we say  $h$  is in the domain of  $\Delta$  and  $\Delta h = f$  if  $h \in \mathcal{F}$  and  $\mathcal{E}(h, g) = -\int fg d\mu$  for every  $g \in \mathcal{F}$ . Let  $\{P_t\}$  be the corresponding semigroup.  $(\mathcal{E}, \mathcal{F})$  is called conservative (or stochastically complete) if  $P_t 1 = 1$  for all  $t > 0$ . Throughout the paper, we assume that  $(\mathcal{E}, \mathcal{F})$  is conservative.

Since  $\mathcal{E}$  is regular,  $\mathcal{E}(f, g)$  can be written in terms of a signed measure  $\Gamma(f, g)$ . To be more precise, for  $f \in \mathcal{F}_b$  (the collection  $\mathcal{F}_b$  is the set of functions in  $\mathcal{F}$  that are essentially bounded)

$\Gamma(f, f)$  is the unique smooth Borel measure (called the energy measure) on  $X$  satisfying

$$\int_X \tilde{g} d\Gamma(f, f) = 2\mathcal{E}(f, fg) - \mathcal{E}(f^2, g), \quad g \in \mathcal{F}_b,$$

where  $\tilde{g}$  is the quasi-continuous modification of  $g \in \mathcal{F}$ . (Recall that  $u : X \rightarrow \mathbb{R}$  is called quasi-continuous if for any  $\varepsilon > 0$ , there exists an open set  $G \subset X$  such that  $\text{Cap}(G) < \varepsilon$  and  $u|_{X \setminus G}$  is continuous. It is known that each  $u \in \mathcal{F}$  admits a quasi-continuous modification  $\tilde{u}$  – see [FOT], Theorem 2.1.3.) Throughout the paper, we will abuse notation and take the quasi-continuous modification of  $g \in \mathcal{F}_b$  without writing  $\tilde{g}$ .  $\Gamma(f, g)$  is defined by

$$\Gamma(f, g) = \frac{1}{2}(\Gamma(f + g, f + g) - \Gamma(f, f) - \Gamma(g, g)), \quad f, g \in \mathcal{F}.$$

$\Gamma(f, g)$  is also local, linear in  $f$  and  $g$ , and satisfies the Leibniz and chain rules – see [FOT], p. 115-116. That is, if  $f_1, \dots, f_m, g$ , and  $\varphi(f_1, \dots, f_m)$  are in  $\mathcal{F}_b$ , and  $\varphi_i$  denotes the partial derivative of  $\varphi$  in the  $i^{\text{th}}$  direction, we have:

$$d\Gamma(fg, h) = f d\Gamma(g, h) + g d\Gamma(f, h),$$

$$d\Gamma(\varphi(f_1, \dots, f_m), g) = \sum_{i=1}^m \varphi_i(f_1, \dots, f_m) d\Gamma(f_i, g).$$

We call  $(X, d, \mu, \mathcal{E})$  a metric measure Dirichlet space, or a MMD space.

**Examples.** 1. If  $M$  is a Riemannian manifold, we can take  $d$  to be the Riemannian metric and  $\mu$  the Riemannian measure. The Dirichlet form  $\mathcal{E}$  is defined by taking its core  $\mathcal{C}$  to be the  $C^\infty$  functions on  $M$  with compact support, and defining

$$\mathcal{E}(f, f) = \int_M |\nabla f|^2 d\mu, \quad f \in \mathcal{C}.$$

The domain  $\mathcal{F}$  of  $\mathcal{E}$  is then the completion of  $\mathcal{C}$  with respect to the norm  $\|f\|_2 + \mathcal{E}(f, f)^{1/2}$ , and  $d\Gamma(f, g) = \nabla f \cdot \nabla g d\mu$ .

2. Cable system of a graph. Given a weighted graph  $(G, E, \nu)$  (see Definition 2.13 below) we can define the *cable system*  $G_C$  by replacing each edge of  $G$  by a copy of  $(0, 1)$ , joined together in the obvious way at the vertices. For further details see [BB5] or [BPY]. Let  $\mu$  be the measure on  $G_C$  given by taking  $d\mu(t) = \nu_{xy} dt$  for  $t$  in the cable connecting  $x$  and  $y$ , where  $\nu_{xy}$  is the conductance of the edge connecting  $x$  and  $y$ ; see [BB5]. One takes as the core  $\mathcal{C}$  the functions in  $C(G_C)$  which have compact support and are  $C^1$  on each cable, and sets

$$\mathcal{E}(f, f) = \int_{G_C} |f'(t)|^2 d\mu(t).$$

One use of this construction is that the restriction to  $G$  of a harmonic function  $h$  on  $G_C$  yields a harmonic function on  $G$ .

3. Let  $D$  be a domain in  $\mathbb{R}^d$  with a smooth boundary. Then let  $\mathcal{C} = C_0^2(\overline{D})$ ,  $\mu$  be Lebesgue measure, and

$$\mathcal{E}(f, f) = \frac{1}{2} \int_D |\nabla f|^2 d\mu.$$

The associated Markov process  $Y$  is Brownian motion on  $D$  with normal reflection on  $\partial D$ . For the extension of this construction to piecewise smooth domains such as the pre-Sierpinski carpet, see [BB4].

4. For fractal sets it is not as easy to describe  $\mathcal{E}$ . However, let  $F \subset \mathbb{R}^d$  be a connected set, and suppose that there exists a geodesic metric  $d$  on  $F$ . Let  $\mu$  be the Hausdorff  $\alpha$ -measure on  $F$  (with respect to  $d$ ) and suppose that

$$c_1 r^\alpha \leq \mu(B_d(x, r)) \leq c_2 r^\alpha, \quad x \in F, r > 0.$$

Let

$$N_{\sigma, \infty}(f) = \sup_{0 < r \leq 1} r^{-\alpha-2\sigma} \int_F \int_F 1_{B(y, r)}(x) |f(x) - f(y)|^2 d\mu(x) d\mu(y),$$

$$\Lambda_{2, \infty}^\sigma(f) = \{u \in L^2(F, \mu) : N_{\sigma, \infty}(f) < \infty\}.$$

There exist many fractals satisfying the above with a Dirichlet form  $\mathcal{E}$  on  $L^2(F, \mu)$  for which the domain  $\mathcal{F}$  of  $\mathcal{E}$  is given by  $\Lambda_{2, \infty}^{\beta/2}$ , and  $c_1 N_{\sigma, \infty}(f) \leq \mathcal{E}(f, f) \leq c_2 N_{\sigma, \infty}(f)$ ; see [Gr2, GHL, Kum2] etc.

In the particular case of the Sierpinski gasket  $F = F_{SG}$ , let  $F_n$  be the set of vertices of triangles of side  $2^{-n}$ ; regard  $F_n$  as a graph with  $x \sim y$  if and only if  $x$  and  $y$  are in some triangle of side  $2^{-n}$ . Then for  $f \in \Lambda_{2, \infty}^{\beta/2}$  with  $\beta = \log 5 / \log 2$ , one has

$$\mathcal{E}(f, f) = c \lim_{n \rightarrow \infty} (5/3)^n \sum_{x \sim y} (f(x) - f(y))^2.$$

In many contexts (including examples 1–3 above) there is a natural metric  $d_{\mathcal{E}}$  associated with the Dirichlet space  $(\mathcal{E}, \mathcal{F}, L^2(M, \mu))$ , defined as follows. Let

$$\mathcal{L}(\lambda) = \{f \in \mathcal{F} : d\Gamma(f, f) \leq \lambda d\mu\}, \tag{2.1}$$

and set

$$d_{\mathcal{E}}(x, y) = \sup\{f(x) - f(y) : f \in \mathcal{L}(1)\}.$$

We cannot use this metric in this paper, since for many true fractal spaces it is known (see [Kus, Hin]) that  $\mathcal{L}(1)$  contains only (quasi-everywhere) constant functions, so that  $d_{\mathcal{E}}(x, y) \equiv 0$ . (One can also have, in other circumstances,  $d_{\mathcal{E}}(x, y) \equiv \infty$  – see e.g. [BS]).

Let  $\beta, \bar{\beta} \geq 2$  and

$$\Psi(s) = \Psi_{\bar{\beta}, \beta}(s) = \begin{cases} s^{\bar{\beta}} & \text{if } s \leq 1 \\ s^{\beta} & \text{if } s > 1. \end{cases}$$

$\Psi(s)$  will give the space/time scaling on the space  $X$ . Generalization of this time scaling factor may be possible (see e.g., [T]), but we do not pursue it here.

We now give a number of conditions which  $X$  may or may not satisfy.

**Definition 2.1.** (a)  $X$  satisfies volume doubling VD if there exists a constant  $c_1$  such that

$$V(x, 2R) \leq c_1 V(x, R) \quad \text{for all } x \in X, R \geq 0. \quad (\text{VD})$$

(b)  $X$  satisfies the *Poincaré inequality* PI( $\Psi$ ) if there exists a constant  $c_2$  such that for any ball  $B = B(x, R) \subset X$  and  $f \in \mathcal{F}$ ,

$$\int_B (f(x) - \bar{f}_B)^2 d\mu(x) \leq c_2 \Psi(R) \int_B d\Gamma(f, f). \quad (\text{PI}(\Psi))$$

Here  $\bar{f}_B = \mu(B)^{-1} \int_B f(x) d\mu(x)$ .

(c) We say a function  $u$  is *harmonic* on a domain  $D$  if  $u \in \mathcal{F}_{loc}$  and  $\mathcal{E}(u, g) = 0$  for all  $g \in \mathcal{F}$  with support in  $D$ . Here  $u \in \mathcal{F}_{loc}$  if and only if for any relatively compact open set  $G$ , there exists a function  $w \in \mathcal{F}$  such that  $u = w$   $\mu$ -a.e. on  $G$ . See page 117 in [FOT] for the definition of  $\mathcal{E}(u, g)$  for  $u \in \mathcal{F}_{loc}$  when  $(\mathcal{E}, \mathcal{F})$  is a regular, strong local Dirichlet form. Functions in  $\mathcal{F}$  are only defined up to quasi-everywhere equivalence; we use a quasi-continuous modification of  $u$ .  $X$  satisfies the *elliptic Harnack inequality* EHI if there exists a constant  $c_3$  such that, for any ball  $B(x, R)$ , whenever  $u$  is a non-negative harmonic function on  $B(x, R)$  then there is a quasi-continuous modification  $\tilde{u}$  of  $u$  that satisfies

$$\sup_{B(x, R/2)} \tilde{u} \leq c_3 \inf_{B(x, R/2)} \tilde{u}. \quad (\text{EHI})$$

Note that by a standard argument (see, e.g., [M1], p. 587) EHI implies that  $\tilde{u}$  is Hölder continuous.

(d) Let  $Q = Q(x_0, T, R) = (0, 4T) \times B(x_0, 2R) =: I \times B_{2R}$ . Let  $u(t, x) : Q \rightarrow \mathbb{R}$ .

- We define  $u_t = \frac{\partial u}{\partial t} \in L^2(dt \times \mu)$  as the derivative in the Schwartz' distribution sense. That is, we define  $u_t$  to be the function  $f$  in  $L^2(dt \times \mu)$  so that for any function  $g : Q \rightarrow \mathbb{R}$  such that  $g(x, \cdot) \in C_K^\infty(0, 4T)$  for each  $x \in B(x_0, 2R)$  and  $g_t = \frac{\partial g}{\partial t} \in L^2(dt \times \mu)$ , then

$$\int_Q (f(x, t)g(x, t) + u(x, t)g_t(x, t)) dt \mu(dx) = 0.$$

- Let  $H(I \rightarrow \mathcal{F}^*)$  be the space of functions  $u \in L^2(I \rightarrow \mathcal{F}^*)$  with the distributional time derivative  $u_t \in L^2(I \rightarrow \mathcal{F}^*)$  equipped with the norm

$$\left( \int_I \|u(t, \cdot)\|_{\mathcal{F}^*}^2 + \|u_t(t, \cdot)\|_{\mathcal{F}^*}^2 dt \right)^{1/2}.$$

Here we identify  $L^2(X, \mu)$  with its own dual and denote the dual of  $\mathcal{F}$  by  $\mathcal{F}^*$ . So,  $\mathcal{F} \subset L^2(X, \mu) \subset \mathcal{F}^*$  with continuous and dense embeddings.

Let  $\mathcal{F}(I \times X) = L^2(I \rightarrow \mathcal{F}) \cap H(I \rightarrow \mathcal{F}^*)$  be a Hilbert space with norm

$$\|u\|_{\mathcal{F}(I \times X)} = \left( \int_I \|u(t, \cdot)\|_{\mathcal{F}}^2 + \|u_t(t, \cdot)\|_{\mathcal{F}^*}^2 dt \right)^{1/2}.$$

- We define  $\mathcal{F}_{loc}(Q)$  to be the set of  $dt \otimes d\mu$ -measurable functions on  $Q$  such that for every relatively compact open set  $D \subset\subset B_{2R}$  and every open interval  $I' \subset\subset I$ , there exists a function  $u' \in \mathcal{F}(I \times X)$  with  $u = u'$  on  $I' \times D$ . We define

$$\mathcal{F}_c(Q) := \{u \in \mathcal{F}(I \times X) : u(t, \cdot) \text{ has compact support in } B_{2R} \text{ for a.e. } t \in I\}.$$

We say a function  $u(t, x) : Q \rightarrow \mathbb{R}$  is a solution of the heat equation in  $Q$  if  $u \in \mathcal{F}_{loc}(Q)$  and

$$\int_J \left[ \int f(t, x) u_t(t, x) \mu(dx) + \mathcal{E}(f(t, \cdot), u(t, \cdot)) \right] dt = 0, \quad \forall J \subset\subset I, \forall f \in \mathcal{F}_c(Q). \quad (2.2)$$

$X$  satisfies the *parabolic Harnack inequality*  $\text{PHI}(\Psi)$ , if there exists a constant  $c_4$  such that the following holds. Let  $x_0 \in X$ ,  $R > 0$ ,  $T = \Psi(R)$ , and  $u = u(t, x)$  be a non-negative solution of the heat equation in  $Q(x_0, T, R)$ . Write  $Q_- = (T, 2T) \times B(x_0, R)$  and  $Q_+ = (3T, 4T) \times B(x_0, R)$ ; then there exists  $\tilde{u} = \tilde{u}(t, x)$  such that  $\tilde{u}(t, \cdot)$  is a quasi-continuous modification of  $u(t, \cdot)$  for each  $t$  and

$$\sup_{Q_-} \tilde{u} \leq c_4 \inf_{Q_+} \tilde{u}. \quad (\text{PHI}(\Psi))$$

Given this PHI, a standard oscillation argument implies that  $\tilde{u}$  is jointly continuous.

**Remark 2.2.** In the case of general MMD spaces we can only define harmonic functions up to quasi-everywhere equivalence. This is why we needed to be careful in our definitions of EHI and  $\text{PHI}(\Psi)$ .

It is easy to deduce from VD that there exist  $c_5, \alpha < \infty$  such that if  $x, y \in X$  and  $0 < r < R$  then

$$\frac{V(x, R)}{V(y, r)} \leq c_5 \left( \frac{d(x, y) + R}{r} \right)^\alpha. \quad (2.3)$$

The following covering lemma holds; cf. [BB5].

**Lemma 2.3.** *Assume that  $X$  satisfies VD. For  $x_0 \in X$  and  $0 < s \leq R \leq \infty$ , there exists a cover of  $B(x_0, R)$  by balls  $B(x_i, s)$  with  $x_i \in B(x_0, R)$  such that no point in  $X$  is in more than  $L_0$  of the  $B(x_i, 2s)$ . Here  $L_0$  depends only on  $s, R$  and the constant  $c_1$  in VD.*

**Definition 2.4.** Let  $A, B$  be disjoint subsets of  $X$ . We define the effective resistance  $R_{\text{eff}}(A, B)$  by

$$R_{\text{eff}}(A, B)^{-1} = \inf \left\{ \int_X d\Gamma(f, f) : f = 0 \text{ on } A \text{ and } f = 1 \text{ on } B, f \in \mathcal{F} \right\}. \quad (2.4)$$

$X$  satisfies the condition  $\text{RES}(\Psi)$  if there exist constants  $c_1, c_2$  such that for any  $x_0 \in X$ ,  $R \geq 0$ ,

$$c_1 \frac{\Psi(R)}{V(x_0, R)} \leq R_{\text{eff}}(B(x_0, R), B(x_0, 2R)^c) \leq c_2 \frac{\Psi(R)}{V(x_0, R)}. \quad (\text{RES}(\Psi))$$

**Definition 2.5.**  $X$  satisfies  $\text{CS}(\Psi)$  if there exist  $\theta \in (0, 1]$  and constants  $c_1, c_2$  such that the following holds. For every  $x_0 \in X$ ,  $R > 0$  there exists a cut-off function  $\varphi (= \varphi_{x_0, R})$  with the properties:

- (a)  $\varphi(x) \geq 1$  for  $x \in B(x_0, R/2)$ .
- (b)  $\varphi(x) = 0$  for  $x \in B(x_0, R)^c$ .
- (c)  $|\varphi(x) - \varphi(y)| \leq c_1(d(x, y)/R)^\theta$  for all  $x, y$ .
- (d) For any ball  $B(x, s)$  with  $0 < s \leq R$  and  $f \in \mathcal{F}$ ,

$$\int_{B(x, s)} f^2 d\Gamma(\varphi, \varphi) \leq c_2(s/R)^{2\theta} \left( \int_{B(x, 2s)} d\Gamma(f, f) + \Psi(s)^{-1} \int_{B(x, 2s)} f^2 d\mu \right). \quad (2.5)$$

**Remarks 2.6.**

1. We call (2.5) a weighted Sobolev inequality. It is clear that to prove (2.5) it is enough to consider nonnegative  $f$ .
2. Suppose  $\text{CS}(\Psi)$  holds for  $X$ , but with (a) above replaced by

$$\varphi(x) \geq 1 \text{ for } x \in B(x_0, \delta R), \quad (2.6)$$

for some  $\delta < \frac{1}{2}$ . Then an easy covering argument (using VD) gives  $\text{CS}(\Psi)$  with  $\delta = \frac{1}{2}$ .

3. Let  $\lambda > 1$ . Suppose that  $\text{CS}(\Psi)$  holds, except that instead of (2.5) we have

$$\int_{B(x, s)} f^2 d\Gamma(\varphi, \varphi) \leq c_2(s/R)^{2\theta} \left( \int_{B(x, \lambda s)} d\Gamma(f, f) + \Psi(s)^{-1} \int_{B(x, \lambda s)} f^2 d\mu \right). \quad (2.7)$$

Then once again it is easy to obtain  $\text{CS}(\Psi)$  with  $\lambda = 2$  by a covering argument.

4. Any operation on the cut-off function  $\varphi$  which reduces  $d\Gamma(\varphi, \varphi)$  while keeping properties (a), (b) and (c) of Definition 2.5 will generate a new cut-off function which still satisfies (2.5).

We can therefore assume that any cut-off function  $\varphi$  satisfies the following:

- (a)  $0 \leq \varphi \leq 1$ .
  - (b) For each  $t \in (0, 1)$  the set  $\{x : \varphi(x) > t\}$  is connected and contains  $B(x_0, R/2)$ .
  - (c) Each connected component  $A$  of  $\{x : \varphi(x) < t\}$  intersects  $B(x_0, R)^c$ .
5. Note that if  $\text{CS}(\Psi)$  holds for  $\Psi = \Psi_{\bar{\beta}, \beta}$ , then  $\text{CS}(\Psi_{\bar{\beta}', \beta'})$  holds if  $\beta' \geq \beta$  and  $\bar{\beta}' \leq \bar{\beta}$ .

We have the following by the same arguments as in Lemma 5.1 of [BB5].

**Lemma 2.7.** *Let  $X$  satisfy VD,  $\text{PI}(\Psi)$  and  $\text{CS}(\Psi)$ . Then  $X$  satisfies  $\text{RES}(\Psi)$ .*

In the definitions which follow, recall that we have a ‘crossover’ from  $t = r^{\bar{\beta}}$  scaling to  $t = r^{\beta}$  scaling at  $r = 1$ . For  $(t, r) \in (0, \infty) \times [0, \infty)$  we consider the two regions:

$$\Lambda_1 = \{(t, r) : t \leq 1 \vee r\}, \quad \Lambda_2 = \{(t, r) : t \geq 1 \vee r\}.$$

Let

$$h_\beta(r, t) = \exp \left( - \left( \frac{r^\beta}{t} \right)^{1/(\beta-1)} \right).$$

**Definition 2.8.** Let  $X$  be a MMD space. We say  $X$  satisfies  $\text{HK}(\Psi)$  if the heat kernel  $p_t(x, y)$  on  $X$  exists and satisfies

$$\frac{c_1 h_{\bar{\beta}}(c_2 d(x, y), t)}{V(x, t^{1/\bar{\beta}})} \leq p_t(x, y) \leq \frac{c_3 h_{\bar{\beta}}(c_4 d(x, y), t)}{V(x, t^{1/\bar{\beta}})}, \quad (2.8)$$

for  $x, y \in X$  and  $t \in (0, \infty)$  with  $(t, d(x, y)) \in \Lambda_1$ , and

$$\frac{c_1 h_{\beta}(c_2 d(x, y), t)}{V(x, t^{1/\beta})} \leq p_t(x, y) \leq \frac{c_3 h_{\beta}(c_4 d(x, y), t)}{V(x, t^{1/\beta})}, \quad (2.9)$$

for  $x, y \in X$  and  $t \in (0, \infty)$  with  $(t, d(x, y)) \in \Lambda_2$ .

**Remark 2.9.**

To understand why the crossover takes the form it does, it is useful to consider the contribution to  $p_t(x, y)$  from various types of paths in  $X$ . Let  $r = d(x, y)$ . First, if  $0 < t \leq 1$  and  $r < 1$  then the behaviour is essentially local.

If  $r \geq t$  then we are in the ‘large deviations’ regime: the main contribution to  $p_t(x, y)$  is from those paths of the Markov process  $Y$  which are within a distance  $O(t/r)$  of a geodesic from  $x$  to  $y$ . So, once the length of the geodesic is given, only the local structure of  $X$  plays a role. Note that in this case the term in the exponential is smaller than  $e^{-ct}$ , so that the volume term  $V(x, t^{1/\bar{\beta}})^{-1}$  could be absorbed into the exponential with a suitable modification of the constants  $c_2$  and  $c_4$ .

Finally, if  $t > 1$  and  $r < t$ , then the paths which contribute to  $p_t(x, y)$  fill out a much larger part of  $X$ : those which lie in  $B(x, t^{1/\beta})$  if  $r < t^{1/\beta}$ , and those which are within a distance  $O(t/r^{\beta-1})$  of a geodesic from  $x$  to  $y$  in the case when  $t^{1/\beta} \leq r \leq t$ .

We will also want to discuss local versions of these conditions. Since these will only depend on  $\Psi(s)$  for  $s \in [0, 1]$ , they are independent of the parameter  $\beta$ . We say  $X$  satisfies  $\text{VD}_{\text{loc}}$  if  $\text{VD}$  holds for  $x \in X$ ,  $0 < R \leq 1$ . Similarly we define  $\text{PI}(\bar{\beta})_{\text{loc}}$ ,  $\text{EHI}_{\text{loc}}$ ,  $\text{CS}(\bar{\beta})_{\text{loc}}$  and  $\text{PHI}(\bar{\beta})_{\text{loc}}$  by requiring the conditions only for  $0 < R \leq 1$ . For  $\text{HK}(\bar{\beta})_{\text{loc}}$  we require the bounds only for  $t \in (0, 1)$  – so only (2.8) is involved. The value 1 here is just for simplicity: each of the local conditions implies an analogous local condition for  $0 < R \leq R_0$  for any (fixed)  $R_0 > 1$  – see Section 2 of [HSC].

Finally, we introduce two local notions which do not include any scaling order.

**Definition 2.10.** (a) We call  $\varphi$  a *cut-off function* for  $A_1 \subset A_2$  if  $\varphi = 1$  on  $A_1$  and is zero on  $A_2^c$ .

(b) We say  $X$  satisfies  $\text{PI}_{\text{loc}}$  if for each  $c_1 > 0$ , there exists  $c_2 > 0$  such that

$$\int_B (f(x) - \bar{f}_B)^2 d\mu(x) \leq c_2 \int_B d\Gamma(f, f) \quad (2.10)$$

for any ball  $B = B(x, c_1) \subset X$  and  $f \in \mathcal{F}$ .

(c) We say  $X$  satisfies  $\text{CC}_{\text{loc}}$  if for every  $x_0 \in X$ , there exists a cut-off function  $\varphi (= \varphi_{x_0})$  for  $B(x_0, 1/2) \subset B(x_0, 1)$  such that

$$\int_{B(x_0, 1)} d\Gamma(\varphi, \varphi) \leq c_3 V(x_0, 1),$$

where  $c_3 > 0$  is independent of  $x_0$  and  $\varphi$ .

$\text{CC}$  stands for ‘controlled cut-off’ functions. Clearly  $\text{PI}(\bar{\beta})_{\text{loc}}$  for any  $\bar{\beta} \geq 2$  implies  $\text{PI}_{\text{loc}}$  and  $\text{CS}(\bar{\beta})_{\text{loc}}$  for any  $\bar{\beta} > 0$  implies  $\text{CC}_{\text{loc}}$ .

We next give some sufficient condition for  $\text{CS}(2)_{\text{loc}}$ .

We say  $(X, d, \mu, \mathcal{E})$  has *regular local cut-off functions* if there exists a constant  $C_1$  such that for all  $x \in X$ ,  $R \in (0, 1)$  there exists a cut-off function  $\varphi : X \rightarrow \mathbb{R}$  for  $B(x, R/2) \subset B(x, R)$  such that  $R\varphi \in \mathcal{L}(C_1)$ , where  $\mathcal{L}(C_1)$  is defined in (2.1).

**Lemma 2.11.** *If the MMD space  $(X, d, \mu, \mathcal{E})$  has regular local cut-off functions such that (c) of Definition 2.5 holds, then it satisfies  $\text{CS}(2)_{\text{loc}}$ .*

*Proof.* Let  $\varphi$  be a regular local cut-off function for  $B(x, R/2) \subset B(x, R)$  such that  $R\varphi \in \mathcal{L}(C_1)$ . Then  $d\Gamma(\varphi, \varphi) \leq C_1^2 R^{-2} d\mu$ , so  $\int_{B(x, s)} f^2 d\Gamma(\varphi, \varphi) \leq C_1^2 R^{-2} \int_{B(x, s)} f^2 d\mu$ . Thus (2.5) holds.  $\square$

If  $d_{\mathcal{E}}$  is a true metric, then  $(X, d_{\mathcal{E}}, \mu, \mathcal{E})$  has regular local cut-off functions since one can take  $C_1 = 2$  and  $\varphi(y) = (1 \wedge (2 - 2R^{-1}d_{\mathcal{E}}(x_0, y))) \vee 0$ . Thus Examples 1–3 above have regular local cut-off functions.

**Definition 2.12.**  $X$  satisfies the condition  $\text{E}(\Psi)$  if for any  $x_0 \in X$ ,  $R \geq 0$ ,

$$c_1 \Psi(R) \leq \mathbb{E}^{x_0} [\tau_{B(x_0, R)}] \leq c_2 \Psi(R), \tag{E(\Psi)}$$

where  $\tau_A = \inf\{t \geq 0 : Y_t \notin A\}$ ,  $Y_t$  is the strong Markov process associated to the Dirichlet form  $(\mathcal{E}, \mathcal{F})$ , and  $\mathbb{E}^{x_0}$  denotes the expectation starting from the point  $x_0$ .  $\text{E}(\Psi)$  describes the ‘walk dimension’ of the associated Markov process.

We also will need to consider weighted graphs.

**Definition 2.13.** Let  $(G, E)$  be an infinite locally finite connected graph. We write  $x \sim y$  if  $(x, y) \in E$ , i.e., there is an edge connecting  $x$  and  $y$ . Define edge weights (conductances)  $\nu_{xy} = \nu_{yx} \geq 0$ ,  $x, y \in G$ , and assume that  $\nu$  is adapted to the graph structure by requiring that  $\nu_{xy} > 0$  if and only if  $x \sim y$ . Let  $\nu_x = \sum_y \nu_{xy}$ , and define a measure  $\nu$  on  $G$  by  $\nu(A) = \sum_{x \in A} \nu_x$ . We call  $(G, \nu)$  a *weighted graph*.

We write  $d(x, y)$  for the graph distance, and define the balls

$$B_G(x, r) = \{y : d(x, y) < r\}.$$

Given  $A \subset G$  write  $\partial A = \{y \in A^c : d(x, y) = 1 \text{ for some } x \in A\}$  for the exterior boundary of  $A$ , and let  $\bar{A} = A \cup \partial A$ .

**Definition 2.14.** A weighted graph  $(G, \nu)$  has *controlled weights* if there exists  $p_0 > 0$  such that for all  $x, y \in G$

$$\frac{\nu_{xy}}{\nu_x} \geq p_0, \quad x \sim y.$$

This was called the  $p_0$ -condition in [GT2].

The Laplacian is defined on  $(G, \nu)$  by

$$\Delta f(x) = \frac{1}{\nu_x} \sum_y \nu_{xy} (f(y) - f(x)).$$

We also define a Dirichlet form  $(\mathcal{E}, \mathcal{F})$  by taking  $\mathcal{F} = L^2(G, \nu)$ , and

$$\mathcal{E}(f, g) = \frac{1}{2} \sum_x \sum_y (f(x) - f(y))(g(x) - g(y)) \nu_{xy}, \quad f, g \in \mathcal{F}.$$

If  $f \in \mathcal{F}$  we define the measure  $\Gamma_G(f, f)$  on  $G$  by setting

$$\Gamma_G(f, f)(x) = \sum_{y \sim x} (f(x) - f(y))^2 \nu_{xy}.$$

The conditions VD, EHI and PHI( $\Psi$ ) for graphs are defined in exactly the same way as for manifolds; see [BB5]. The definitions of PI( $\Psi$ ) and RES( $\Psi$ ) are also the same. For the bound HK( $\Psi$ ) we only require (2.9). The condition CS( $\Psi$ ) is also the same; the weighted Sobolev inequality (2.5) takes the form

$$\begin{aligned} \sum_{x \in B_G(x_1, s)} f(x)^2 \Gamma_G(\varphi, \varphi)(x) & \tag{2.11} \\ & \leq c_2 \left(\frac{s}{R}\right)^{2\theta} \left( \sum_{x \in B_G(x_1, 2s)} \Gamma_G(f, f)(x) + \Psi(s)^{-1} \sum_{x \in B_G(x_1, 2s)} \nu_x f(x)^2 \right). \end{aligned}$$

It is easy to check that  $\text{PI}_{\text{loc}}$  and  $\text{CC}_{\text{loc}}$  hold for any weighted graph with controlled weights. In fact,  $\text{PI}(\bar{\beta})_{\text{loc}}$  and  $\text{CS}(\bar{\beta})_{\text{loc}}$  hold for any choice of  $\bar{\beta} \geq 2$  on such graphs, since it is irrelevant to treat  $R < 1$  for graphs.

We summarize the conditions we have introduced:

VD	Volume doubling
PI( $\Psi$ )	Poincaré inequality
EHI	Elliptic Harnack inequality
PHI( $\Psi$ )	Parabolic Harnack inequality
RES( $\Psi$ )	Resistance exponent
CS( $\Psi$ )	Cut-off Sobolev inequality
CC	Controlled cut-off functions
HK( $\Psi$ )	Heat kernel estimates
E( $\Psi$ )	Walk dimension

We will need the following:

**Theorem 2.15.** (See [BBK], [HSC] Theorem 5.3, [GT3]). The following are equivalent:

- (a)  $X$  satisfies  $PHI(\Psi)$ .
- (b)  $X$  satisfies  $HK(\Psi)$ .
- (c)  $X$  satisfies  $VD$ ,  $EHI$  and  $RES(\Psi)$ .
- (d)  $X$  satisfies  $VD$ ,  $EHI$  and  $E(\Psi)$ .

*Proof.* The equivalence of (a) and (b) is given in [BBK]. (See also [HSC] for the case where solutions to the heat equation are sufficient regular.) That these are equivalent to (c) and (d) is proved in [GT3]. (See [GT2] for the graph case.)  $\square$

The first of our main theorems is the following. (The graph case was proved in [BB5]).

**Theorem 2.16.** Suppose that  $X$  is either an infinite connected weighted graph with controlled weights, or a MMD space. The following are equivalent:

- (a)  $X$  satisfies  $VD$ ,  $PI(\Psi)$  and  $CS(\Psi)$ .
- (b)  $X$  satisfies  $PHI(\Psi)$ .

Our second topic is the stability of  $PHI(\Psi)$ . We will actually discuss two kinds of stability.

**Definition 2.17.** A property  $P$  is *stable under bounded perturbation* if whenever  $P$  holds for  $(\mathcal{E}^{(1)}, \mathcal{F})$ , then it holds for  $(\mathcal{E}^{(2)}, \mathcal{F})$ , provided

$$c_1 \mathcal{E}^{(1)}(f, f) \leq \mathcal{E}^{(2)}(f, f) \leq c_2 \mathcal{E}^{(1)}(f, f), \quad \text{for all } f \in \mathcal{F}. \quad (2.12)$$

The following result is due to Le Jan ([LJ], Proposition 1.5.5(b)). A simple proof is given in [Mos] p. 389.

**Lemma 2.18.** Let  $X$  be a MMD space. Suppose  $(\mathcal{E}^{(i)}, \mathcal{F})$ ,  $i = 1, 2$  are strong local regular Dirichlet forms that satisfy (2.12). Then the energy measures  $\Gamma^{(i)}$  satisfy

$$c_1 d\Gamma^{(1)}(f, f) \leq d\Gamma^{(2)}(f, f) \leq c_2 d\Gamma^{(1)}(f, f), \quad \text{for all } f \in \mathcal{F}. \quad (2.13)$$

It is immediate from Lemma 2.18 that the conditions  $PI(\Psi)$  and  $CS(\Psi)$  are stable under bounded perturbations. So we deduce:

**Theorem 2.19.** Let  $X$  be a MMD space. Then  $PHI(\Psi)$  and  $HK(\Psi)$  are stable under bounded perturbations.

The second kind of stability is stability under rough isometries.

**Definition 2.20.** For each  $i = 1, 2$ , let  $(X_i, d_i, \mu_i)$  be either a metric measure space or a weighted graph. A map  $\varphi : X_1 \rightarrow X_2$  is a *rough isometry* if there exist constants  $c_1 > 0$  and  $c_2, c_3 > 1$  such that

$$X_2 = \bigcup_{x \in X_1} B_{d_2}(\varphi(x), c_1),$$

$$c_2^{-1}(d_1(x, y) - c_1) \leq d_2(\varphi(x), \varphi(y)) \leq c_2(d_1(x, y) + c_1),$$

and

$$c_3^{-1} \mu_1(B_{d_1}(x, c_1)) \leq \mu_2(B_{d_2}(\varphi(x), c_1)) \leq c_3 \mu_1(B_{d_1}(x, c_1)).$$

If there exists a rough isometry between two spaces they are said to be *roughly isometric*. (One can check this is an equivalence relation.)

This concept was introduced by Kanai in [Kan1, Kan2]. A rough isometry between  $X_1$  and  $X_2$  means that the global structure of the two spaces is the same. However, to have stability of Harnack inequalities, we also require some control over the local structure. In the case of graphs it is enough to have controlled weights, but for metric measure spaces more regularity is needed. (In [Kan1, Kan2] this local control was obtained by geometrical assumptions on the manifolds).

The following theorem concerns the stability of  $\text{PHI}(\Psi)$  under rough isometries.

**Theorem 2.21.** *Let  $X_i$  be either a MMD space satisfying  $\text{VD}_{\text{loc}}$  and  $\text{PI}_{\text{loc}}$  or a graph with controlled weights, and suppose there exists a rough isometry  $\varphi : X_1 \rightarrow X_2$ . Let  $\Psi_i(s) = s^{\bar{\beta}_i} 1_{\{s \leq 1\}} + s^{\beta} 1_{\{s \geq 1\}}$ .*

(a) *Suppose that  $X_2$  satisfies  $\text{PI}(\bar{\beta}_2)_{\text{loc}}$ . If  $X_1$  satisfies  $\text{VD}$ ,  $\text{CC}_{\text{loc}}$  and  $\text{PI}(\Psi_1)$  then  $X_2$  satisfies  $\text{VD}$  and  $\text{PI}(\Psi_2)$ .*

(b) *Suppose that  $X_2$  satisfies  $\text{CS}(\bar{\beta}_2)_{\text{loc}}$ . If  $X_1$  satisfies  $\text{VD}$  and  $\text{CS}(\Psi_1)$  then  $X_2$  satisfies  $\text{VD}$  and  $\text{CS}(\Psi_2)$ .*

By this theorem together with Theorem 2.16, we see that  $\text{PHI}(\Psi)$  is stable under rough isometries, given suitable local regularity of the two spaces.

### 3. Construction of cut-off functions.

In this section we will prove Theorem 2.16 (b)  $\Rightarrow$  (a). If  $X$  satisfies  $\text{PHI}(\Psi)$ , then (VD) and  $\text{HK}(\Psi)$  hold by Theorem 2.15.

**Lemma 3.1.**  *$\text{PI}(\Psi)$  holds.*

*Proof.* Let  $\bar{Y}$  be the process  $Y$  'reflected at the boundary' of the ball  $B = B(x_0, R) \subset X$ : see [Ch] for this construction in a general context. Write  $\bar{p}_t(x, y)$  for the transition density of  $\bar{Y}$ , and  $p_t^0(x, y)$  for that of the process  $Y$  killed on exiting  $B$ .

The argument of [SC1] gives  $\text{PI}(\Psi)$  once we have the estimate

$$\bar{p}_t(x, y) \geq p_t^0(x, y) \geq cV(x_0, R)^{-1}, \quad x, y \in B(x_0, R/2), \frac{1}{2}\Psi(R) \leq t \leq \Psi(R).$$

This is proved in [BBK], as a key step in the derivation of  $\text{HK}(\Psi)$  from  $\text{PHI}(\Psi)$ .  $\square$

We now prove that  $\text{PHI}(\Psi) \Rightarrow \text{CS}(\Psi)$ , and follow the arguments in [BB5] and [B2]. The main difference from [BB5] is that a strong transience condition (called (FVG)) was needed there in the initial arguments. We will also slightly simplify the arguments in [B2].

Let  $D = B(x_0, R - \varepsilon)$  where  $\varepsilon < R/10$ , and  $\lambda > 0$ . Let  $Y$  be the process associated with the Dirichlet form  $(\mathcal{E}, \mathcal{F})$ . Let  $G_\lambda^D$  be the resolvent associated with the process  $Y$  killed on exiting  $D$ ; that is,

$$G_\lambda^D f(x) = \mathbb{E}^x \int_0^{\tau_D} e^{-\lambda t} f(Y_t) dt,$$

for bounded measurable  $f$ , where  $\tau_D = \inf\{t : Y_t \in X - D\}$ . Let  $p_t^D(\cdot, \cdot)$  be the heat kernel of  $Y$  killed on exiting  $D$ . Then the Green kernel of  $G_\lambda^D$  is given by

$$g_\lambda^D(x, y) = \int_0^\infty e^{-\lambda t} p_t^D(x, y) dt.$$

We use the Green kernel to build a cut-off function  $\varphi$ .

**Lemma 3.2.** *Let  $x_0 \in X$ . Then there exists  $\delta > 0$  such that if  $\lambda = c_0 \Psi(R)^{-1}$*

$$g_\lambda^D(x_0, y) \leq C_1 \frac{\Psi(R)}{V(x_0, R)}, \quad y \in B(x_0, \delta R)^c, \quad (3.1)$$

$$g_\lambda^D(x_0, y) \geq C_2 \frac{\Psi(R)}{V(x_0, R)}, \quad y \in B(x_0, \delta R). \quad (3.2)$$

*Proof.* This follows easily from HK( $\Psi$ ) by integration. □

**Lemma 3.3.** *Let  $x_0$  and  $R$  be as above, and let  $x, y \in B(x_0, \delta R)^c$ . Then there exists  $\theta > 0$  such that*

$$|g_\lambda^D(x_0, x) - g_\lambda^D(x_0, y)| \leq c_1 \left( \frac{d(x, y)}{R} \right)^\theta \sup_{B(x_0, \delta R)^c} g_\lambda^D(x_0, \cdot). \quad (3.3)$$

*Proof.* The Hölder continuity of  $p_t^D$  follows from PHI( $\Psi$ ) by a standard argument; see [M2]. Integrating we obtain (3.3). □

Fix  $x_0 \in X$  and let  $B' = B(x_0, \delta R)$ ,  $B = B(x_0, R)$ ,  $D = B(x_0, R - \varepsilon)$  where  $\varepsilon < R/10$ . Let  $\lambda = c_0 \Psi(R)^{-1}$  and define

$$\varphi(x) = 1 \wedge (c \Psi(R)^{-1} G_\lambda^D 1_{B'}(x)),$$

where  $c$  is chosen so that  $\varphi(x) = 1$  on  $x \in B'$ . Using Lemmas 3.2 and 3.3, it is easy to check that  $\varphi$  is a cut-off function for  $B' \subset B$  that satisfies Definition 2.5 (a)–(c). To complete the proof of CS( $\Psi$ ), we need to establish (2.5).

**Proposition 3.4.** Let  $x_1 \in X$  and  $f \in \mathcal{F}$ . Let  $\delta$  be defined by Lemma 3.2 and let  $I = B(x_1, \delta s)$  with  $0 < s \leq R$  and  $I^* = B(x_1, s)$ . There exist  $c_1, c_2 > 0$  such that for all  $f \in \mathcal{F}$ ,

$$\int_I f^2 d\Gamma(\varphi, \varphi) \leq c_1 (s/R)^{2\theta} \left( \int_{I^*} d\Gamma(f, f) + c_2 \Psi(s)^{-1} \int_{I^*} f^2 d\mu \right). \quad (3.4)$$

*Proof. Case 1.* We first consider the case where  $s = R$  and  $x_1 = x_0$ . Let

$$\mathcal{F}_D = \{f \in \mathcal{F} : \tilde{f} = 0 \text{ q.e. on } X - D\}.$$

Set

$$\mathcal{E}_\lambda(f, g) = \mathcal{E}(f, g) + \lambda \int fg d\mu.$$

Let  $v = G_\lambda^D 1_{B'}$ . Note that

$$v(x) \leq \int_{B'} g^D(x, y) d\mu(y) \leq \mathbb{E}^x[\tau_D] \leq c\Psi(R), \quad x \in D, \quad (3.5)$$

by Theorem 2.15. By [FOT] Theorem 4.4.1,  $v \in \mathcal{F}_D$  and is quasi-continuous. Further, since  $Y$  is continuous,  $v = 0$  on  $\overline{D}^c$ . Let  $f \in \mathcal{F}$ . Then

$$\int_B f^2 d\Gamma(v, v) \leq \int_X f^2 d\Gamma(v, v) = \int_X d\Gamma(f^2 v, v) - \int_X 2fv d\Gamma(f, v).$$

Since  $v \in \mathcal{F}_D$  we have  $f^2 v \in \mathcal{F}_D$ , so by [FOT], Theorem 4.4.1,

$$\int_X d\Gamma(f^2 v, v) = \mathcal{E}(f^2 v, G_\lambda^D 1_{B'}) \leq \mathcal{E}_\lambda(f^2 v, G_\lambda^D 1_{B'}) = \int_X f^2 v 1_{B'} d\mu \leq c\Psi(R) \int_{B'} f^2 d\mu,$$

where we used (3.5) in the last inequality. Using Cauchy-Schwarz and (3.5), we obtain

$$\begin{aligned} \left| \int_X 2fv d\Gamma(f, v) \right| &\leq c \left( \int_X v^2 d\Gamma(f, f) \right)^{1/2} \left( \int_X f^2 d\Gamma(v, v) \right)^{1/2} \\ &\leq c\Psi(R) \left( \int_B d\Gamma(f, f) \right)^{1/2} \left( \int_X f^2 d\Gamma(v, v) \right)^{1/2}. \end{aligned}$$

So, writing  $H = \int_X f^2 d\Gamma(v, v)$ ,  $J = \int_B d\Gamma(f, f)$ ,  $K = \int_B f^2 d\mu$ , we have

$$H \leq c\Psi(R)K + c\Psi(R)J^{1/2}H^{1/2},$$

from which it follows that  $H \leq c\Psi(R)K + c\Psi(R)^2 J$ . From this, (3.4) with  $s = R$  follows easily.

**Case 2.** Define

$$Q(b) = Q(x_0, b) = \{y : g_\lambda^D(x_0, y) > b\}.$$

and let

$$h = C_2 \Psi(R) / (2V(x_0, R)),$$

where  $C_2$  is as in Lemma 3.2. Note that by Lemma 3.2 and the fact  $g_\lambda^D(x_0, y) = 0$  for  $y \notin D$ ,

$$B(x_0, \delta R) \subset Q(2h) \subset Q(h) \subset B(x_0, R).$$

In Case 2, we will consider the situation that either

$$I^* \subset Q(2h) \tag{3.6}$$

or

$$I^* \cap B(x_0, \delta R/2) = \emptyset \tag{3.7}$$

hold. Since  $\varphi \equiv 1$  on  $Q(2h)$ , (3.4) is clear if (3.6) holds. Thus, we consider when (3.7) holds. Let  $\psi_s(x) = 1 \wedge (c\Psi(s)^{-1}G_\lambda^{B(x_0, s-\epsilon)}1_I(x))$  be a cut-off function for  $I \subset I^*$  given by Case 1. Let  $\varphi_0(x) = \Psi(R)^{-1}G_\lambda^D 1_{B''}(x)$  where  $B'' = B(x_0, \delta R/2)$  and  $\varphi_1(x) = \varphi_0(x) - \min_{y \in I^*} \varphi(y)$ , then by Lemma 3.3,

$$\varphi_1(x) \leq c(s/R)^\theta = L, \quad x \in I^*.$$

Let

$$\begin{aligned} A &= \int_I f^2 d\Gamma(\varphi, \varphi), \\ D &= \int_{I^*} d\Gamma(f, f) + \Psi(s)^{-1} \int_{I^*} f^2, \\ F &= \int_{I^*} f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_1). \end{aligned}$$

Now as

$$d\Gamma(f^2 \psi_s^2 \varphi, \varphi) \leq d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0) = f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_0) + \varphi_1 d\Gamma(f^2 \psi_s^2, \varphi_0),$$

we have

$$A \leq F = \int_{I^*} f^2 \psi_s^2 d\Gamma(\varphi_1, \varphi_0) = \int_{I^*} d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0) - \int_{I^*} \varphi_1 d\Gamma(f^2 \psi_s^2, \varphi_0). \tag{3.8}$$

For the first term in (3.8)

$$\begin{aligned} \int_{I^*} d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0) &= \int_X d\Gamma(f^2 \psi_s^2 \varphi_1, \varphi_0) \\ &= \mathcal{E}_\lambda(f^2 \psi_s^2 \varphi_1, \Psi(R)^{-1}G_\lambda^D 1_{B''}) - \lambda \int_X f^2 \psi_s^2 \varphi_1 \varphi_0 d\mu \\ &\leq \mathcal{E}_\lambda(f^2 \psi_s^2 \varphi_1, \Psi(R)^{-1}G_\lambda^D 1_{B''}) = \Psi(R)^{-1} \int_{B''} f^2 \psi_s^2 \varphi_1 d\mu = 0. \end{aligned}$$

Here we used the fact that  $\varphi_1 \geq 0$  on  $I^*$  and that the support of  $\psi_s$  is in  $I^*$ , hence outside  $B''$  (due to (3.7)).

The final term in (3.8) is handled, using the Leibniz and chain rules and Cauchy-Schwarz, as

$$\begin{aligned} \left| \int_{I^*} \varphi_1 d\Gamma(f^2 \psi_s^2, \varphi_0) \right| &\leq 2 \left| \int_{I^*} \varphi_1 f \psi_s^2 d\Gamma(f, \varphi_0) \right| + 2 \left| \int_{I^*} \varphi_1 f^2 \psi_s d\Gamma(\psi_s, \varphi_0) \right| \\ &\leq c \left\{ \left( \int_{I^*} \psi_s^2 d\Gamma(f, f) \right)^{1/2} + \left( \int_{I^*} f^2 d\Gamma(\psi_s, \psi_s) \right)^{1/2} \right\} \left( \int_{I^*} \varphi_1^2 f^2 \psi_s^2 d\Gamma(\varphi_0, \varphi_0) \right)^{1/2} \\ &\leq cD^{1/2}LF^{1/2}, \end{aligned}$$

where we used Case 1 in the final line. Thus we obtain  $A \leq F \leq cDL^2$  so that (3.4) holds.

**Case 3.** We finally consider the general case. When either (3.6) or (3.7) holds, the result is already proved in Case 2. So assume that neither of them hold. Then  $I^*$  must intersect both  $B(x_0, \delta R/2)$  and  $B(x_0, \delta R)^c$ , so  $s \geq \delta R/4$ . We use Lemma 2.3 to cover  $I$  with balls  $B_i = B(x_i, c_1 R)$ , where  $c_1 \in (0, \delta/4)$  has been chosen small enough so that each  $B_i^* := B(x_i, c_1 R/\delta)$  satisfies at least one of (3.6) or (3.7). We can then apply (3.4) with  $I$  replaced by each ball  $B_i$ : writing  $s' = c_1 R$  we have

$$\int_{B_i} f^2 d\Gamma(\varphi, \varphi) \leq c_2 (s'/R)^{2\theta} \left( \int_{B_i^*} d\Gamma(f, f) + \Psi(s')^{-1} \int_{B_i^*} f^2 d\mu \right).$$

We then sum over  $i$ . Since no point of  $I^*$  is in more than  $L_0$  (not depending on  $x_0$  or  $R$ ) of the  $B_i^*$ , and  $s/c_1 \leq s' \leq s$ , we obtain (3.4) for  $I$ .  $\square$

#### 4. Sobolev inequalities and elliptic Harnack inequality.

In this section we will prove Theorem 2.16 (a)  $\Rightarrow$  (b). Assume that  $X$  satisfies VD, PI( $\Psi$ ) and CS( $\Psi$ ). Using Theorem 2.15 (c)  $\Rightarrow$  (a) and Lemma 2.7, it is enough to show VD + PI( $\Psi$ ) + CS( $\Psi$ )  $\Rightarrow$  EHI. For  $x \in X$ ,  $R \geq 0$  let  $\varphi = \varphi_{x,R}$  be a cut-off function given by CS( $\Psi$ ). We define the measure  $\gamma = \gamma_{x,R}$  by

$$d\gamma = d\mu + \Psi(R)d\Gamma(\varphi, \varphi).$$

We remark that we do not know if the measure  $\gamma$  satisfies volume doubling. The first step in the argument is to use CS( $\Psi$ ) to obtain a weighted Sobolev inequality. For any set  $J \subset X$  set

$$J^s = \{y : d(y, J) \leq s\}.$$

**Proposition 4.1.** *Let  $s \leq R$  and  $J \subset B(x_0, R)$  be a finite union of balls of radius  $s$ . There exist  $\kappa > 1$  and  $c_1 > 0$  such that*

$$\left( \mu(J)^{-1} \int_J |f|^{2\kappa} d\gamma \right)^{1/\kappa} \leq c_1 \left( \Psi(R)\mu(J)^{-1} \int_{J^s} d\Gamma(f, f) + (s/R)^{-2\theta} \mu(J)^{-1} \int_J f^2 d\gamma \right). \quad (4.1)$$

We omit the proof, since it is the same as that of Theorem 5.4 of [BB5].

The next result is the generalization of Lemma 4 of [M1] to the case of a MMD space.

**Lemma 4.2.** *Let  $D$  be a domain in  $X$ , let  $u$  be positive and harmonic in  $D$ ,  $v = u^k$ , where  $k \in \mathbb{R}$ ,  $k \neq \frac{1}{2}$ , and let  $\eta$  be supported in  $D$ . Suppose  $\int_D d\Gamma(\eta, \eta) < \infty$ , then*

$$\int_D \eta^2 d\Gamma(v, v) \leq \left( \frac{2k}{2k-1} \right)^2 \int_D v^2 d\Gamma(\eta, \eta).$$

*Proof.* Let  $g \in \mathcal{F}$  be supported by  $D$ . Then if  $u' = Gh$  where  $h = 0$  on  $D$  we have

$$\int_D d\Gamma(gu', u') = \int_X d\Gamma(gu', u') = \int_X gu'h d\mu = 0.$$

Hence, approximating  $u$  by functions of the form  $u'$  we deduce that

$$\int_D d\Gamma(gu, u) = 0.$$

Using this, and taking  $g = \eta^2 k^2 u^{2k-2}$ , we conclude that

$$\int_D \eta^2 d\Gamma(v, v) = \int_D g d\Gamma(u, u) = - \int_D u d\Gamma(g, u). \quad (4.2)$$

Using the Leibniz and chain rules, the right hand side is equal to

$$-2k \int_D \eta v d\Gamma(\eta, v) - (2k-2) \int_D \eta^2 d\Gamma(v, v).$$

Thus,

$$\begin{aligned} \int_D \eta^2 d\Gamma(v, v) &= -\frac{2k}{2k-1} \int_D \eta v d\Gamma(v, \eta) \\ &\leq \frac{2|k|}{|2k-1|} \left( \int_D \eta^2 d\Gamma(v, v) \right)^{1/2} \left( \int_D v^2 d\Gamma(\eta, \eta) \right)^{1/2}, \end{aligned}$$

where we used Cauchy-Schwarz. Dividing and squaring, we obtain the result.  $\square$

Let  $u$  be harmonic and nonnegative in  $B(x_0, 4R)$ . By looking at  $u + \varepsilon$  and letting  $\varepsilon \downarrow 0$  we may without loss of generality suppose  $u$  is strictly positive. Note that, as for a general MMD space we do not initially have any *a priori* continuity for  $u$ , we do not obtain a pointwise bound in (4.3).

**Proposition 4.3.** *Let  $v$  be either  $u$  or  $u^{-1}$ . There exists  $c_1$  such that if  $B(x, 2r) \subset B(x_0, 4R)$  and  $0 < q < 2$ , then*

$$\text{ess sup}_{B(x, r/2)} v^{2q} \leq c_1 V(x, 2r)^{-1} \int_{B(x, 2r)} \left( \Psi(r) d\Gamma(v^q, v^q) + v^{2q} d\mu \right). \quad (4.3)$$

*Proof.* Let  $\varphi_0$  be a (regularized) cut-off function given by  $\text{CS}(\Psi)$  for  $B(x, r)$ . Let  $h_n = 1 - 2^{-n}$ ,  $0 \leq n \leq \infty$ , so that  $0 = h_0 < h_\infty = 1$ . For  $k \geq 0$  set

$$\varphi_k(x) = (\varphi_0(x) - h_k)^+, \quad d\gamma_0 = d\mu + \Psi(r)d\Gamma(\varphi_0, \varphi_0).$$

Set  $A_k = \{x : \varphi_0(x) > h_k\}$ , and note that  $B(x, r/2) \subset A_{n_0} \subset A_0 \subset B(x, r)$  for every  $n_0$ . We therefore have, writing  $V$  for  $V(x, r)$ ,

$$c_2V \leq \mu(A_k) \leq V, \quad k \geq 0.$$

The Hölder condition on  $\varphi_0$  given by  $\text{CS}(\Psi)$  implies that if  $x \in A_{k+1}$  and  $y \in A_k^c$ , then  $d(x, y) \geq c_3r2^{-k/\theta}$ . Set  $s_k = \frac{1}{2}c_3r2^{-k/\theta}$ , and note that  $\varphi_k > c_42^{-k}$  on  $A_{k+1}^{s_k}$ . Let  $\{B_i\}$  be a cover of  $A_{k+1}$  by balls of radius  $s_k/2$ , and let  $J_{k+1} = \cup_i B_i$ . Write  $J'_{k+1} = J_{k+1}^{s_k/2}$ ,  $A'_{k+1} = A_{k+1}^{s_k}$  and note that  $A_{k+1} \subset J_{k+1} \subset J'_{k+1} \subset A'_{k+1}$ .

From Proposition 4.1 with  $f = v^p$  and  $s$  replaced by  $s_k/2$ ,

$$\begin{aligned} \left(V^{-1} \int_{A_{k+1}} f^{2\kappa} d\gamma_0\right)^{1/\kappa} &\leq \left(V^{-1} \int_{J_{k+1}} f^{2\kappa} d\gamma_0\right)^{1/\kappa} \\ &\leq c_5 V^{-1} \left[ \Psi(r) \int_{J'_{k+1}} d\Gamma(f, f) + (r/s_k)^{2\theta} \int_{J'_{k+1}} f^2 d\gamma_0 \right] \\ &\leq c_6 V^{-1} \left[ \Psi(r) \int_{A'_{k+1}} d\Gamma(f, f) + 2^{2k} \int_{A_k} f^2 d\gamma_0 \right]. \end{aligned} \quad (4.4)$$

By Lemma 4.2, we have the ‘converse to the Poincaré inequality’ for  $f = v^p$ , which controls the first term in (4.4).

$$\begin{aligned} \Psi(r) \int_{A'_{k+1}} d\Gamma(f, f) &\leq \Psi(r)(c_7 2^{-k})^{-2} \int_{A'_{k+1}} \varphi_k^2 d\Gamma(f, f) \leq c_8 2^{2k} \Psi(r) \int_{A_k} \varphi_k^2 d\Gamma(f, f) \\ &\leq c_9 2^{2k} \Psi(r) \left(\frac{2p}{2p-1}\right)^2 \int_{A_k} f^2 d\Gamma(\varphi_k, \varphi_k) \leq c_{10} 2^{2k} \left(\frac{2p}{2p-1}\right)^2 \int_{A_k} f^2 d\gamma_0. \end{aligned}$$

We therefore deduce that

$$\left(V^{-1} \int_{A_{k+1}} f^{2\kappa} d\gamma_0\right)^{1/\kappa} \leq c_{11} \left(\frac{2p}{2p-1}\right)^2 2^{2k} V^{-1} \int_{A_k} f^2 d\gamma_0. \quad (4.5)$$

Now, similarly to the first part of Moser’s argument [M1] with  $p_n = q\kappa^n$  for appropriate  $q$ , we have

$$\text{ess sup}_{B(x, r/2)} v \leq c_{12} \left(V^{-1} \int_{B(x, r)} v^{2q} d\gamma_0\right)^{1/(2q)}.$$

Using  $\text{CS}(\Psi)$  and VD, we obtain (4.3) – see [BB5] Proposition 5.8 for details.  $\square$

Recall that  $\varphi$  is a cut-off function for  $B(x_0, R)$  given by  $\text{CS}(\Psi)$ . We define

$$Q(t) = \{x : \varphi(x) > t\}, \quad 0 < t < 1,$$

and write  $Q(1)$  for the interior of  $\{x : \varphi(x) \geq 1\}$ .

Using Proposition 4.3, we have the following. See [BB5] Corollary 5.9 for the proof.

**Corollary 4.4.** *Let  $1 > s > t > 0$ . There exists  $\zeta > 2$  such that if  $0 < q < \frac{1}{3}$ ,*

$$\text{ess sup}_{Q(s)} v^{2q} \leq c_1 (s-t)^{-\zeta} V(x_0, R)^{-1} \int_{Q(t)} v^{2q} d\gamma. \quad (4.6)$$

Now our goal is to deduce the elliptic Harnack inequality. The following corresponds to the second part of Moser's arguments.

Let  $w = \log u$ , and write  $\bar{w} = V(x_0, R)^{-1} \int_{B(x_0, R)} w d\mu$ .

**Proposition 4.5.** *(cf. [BB5] Proposition 5.7, Corollary 5.10.)*

(a) There exists  $c_1$  such that

$$\int_{B(x_0, 2R)} d\Gamma(w, w) \leq c_1 \frac{V(x_0, R)}{\Psi(R)}.$$

(b) Let  $1 \geq s > t > 0$ . Then

$$\int_{\{|w - \bar{w}| > A\} \cap Q(s)} d\gamma \leq c_2 \frac{V(x_0, R)}{A^2}.$$

*Proof.* Again, this is essentially Moser's proof. Let  $\varphi_1(x)$  be a cut-off function given by  $\text{CS}(\Psi)$  for the ball  $B^* := B(x_0, 4R)$ . So

$$\int_{B(x_0, 2R)} d\Gamma(w, w) \leq c \int_{B^*} \varphi_1^2 d\Gamma(w, w).$$

Applying (4.2) with  $\eta = \varphi_1$ ,  $v = w$ ,  $g = \varphi_1^2/u^2$  and  $D = B^*$ , we have

$$\int_{B^*} \varphi_1^2 d\Gamma(w, w) = - \int_{B^*} u d\Gamma(\varphi_1^2/u^2, u).$$

Using the Leibniz and chain rules, the right hand side is equal to

$$-2 \int_{B^*} \varphi_1 d\Gamma(\varphi_1, w) + 2 \int_{B^*} \varphi_1^2 d\Gamma(w, w).$$

Thus,

$$\int_{B^*} \varphi_1^2 d\Gamma(w, w) = 2 \int_{B^*} \varphi_1 d\Gamma(\varphi_1, w) \leq 2 \left( \int_{B^*} d\Gamma(\varphi_1, \varphi_1) \right)^{1/2} \left( \int_{B^*} \varphi_1^2 d\Gamma(w, w) \right)^{1/2},$$

where we used Cauchy-Schwarz. Dividing and squaring,

$$\int_{B^*} \varphi_1^2 d\Gamma(w, w) \leq 4 \int_{B^*} d\Gamma(\varphi_1, \varphi_1).$$

Finally, using  $\text{CS}(\Psi)$  in  $B^*$  with  $f \in \mathcal{F}$  such that  $f|_{B(x_0, 8R)} \equiv 1$  (since  $(\mathcal{E}, \mathcal{F})$  is a regular Dirichlet form, such an  $f$  exists) and VD we deduce that

$$\int_{B^*} d\Gamma(\varphi_1, \varphi_1) \leq c\Psi(R)^{-1}V(x_0, R).$$

The proof of (b) is the same as that of [BB5] Corollary 5.10, so we omit it.  $\square$

In order to get the Harnack inequality the argument in [M2] required a generalization of the John-Nirenberg inequality with a complicated proof. Bombieri [Bom] found a way to avoid such an argument for elliptic second order differential equations. Moser (Lemma 3 in [M3]) carried the idea over to the parabolic case and Bombieri and Giusti (Theorem 4 in [BG]) obtained the inequality in an abstract setting. (See also Lemma 2.2.6 in [SC2].) This argument can be applied to our setting (with suitable modifications) and we can show that Corollary 4.4 and Proposition 4.5 (b) give

$$\text{ess sup}_{B(x_0, R/2)} \log u \leq c_1,$$

(see [BB5] Lemma 5.11 and Theorem 5.12 for a detailed proof). Let  $v = u^{-1}$ . The same argument implies  $\text{ess sup}_{B(x_0, R/2)} \log v \leq c_1$ , or  $\text{ess inf}_{B(x_0, R/2)} \log u \geq -c_1$ . Combining we deduce

$$e^{-c_1} \leq \text{ess inf}_{B(x_0, R/2)} u \leq \text{ess sup}_{B(x_0, R/2)} u \leq e^{c_1}.$$

We thus obtain the following.

**Theorem 4.6.** *There exists  $c_1$  such that if  $u$  is nonnegative and harmonic in  $B(x_0, 4R)$ , then*

$$\text{ess sup}_{B(x_0, R/2)} u \leq c_1 \text{ess inf}_{B(x_0, R/2)} u.$$

**Proof of Theorem 2.16 (a)  $\Rightarrow$  (b).**

As we mentioned in the beginning of this section, using Theorem 2.15 (c)  $\Rightarrow$  (a) and Lemma 2.7, it is enough to show  $\text{VD} + \text{PI}(\Psi) + \text{CS}(\Psi) \Rightarrow \text{EHI}$ .

Let  $u$  be nonnegative and harmonic in  $B(x_0, 4R)$ . Suppose  $x_1$  and  $r$  are such that  $B(x_1, 3r) \subset B(x_0, 4R)$ . By looking at  $Cu + D$  for suitable constants  $C$  and  $D$ , we may suppose that  $\text{ess sup}_{B(x_1, 2r)} u = 1$  and  $\text{ess inf}_{B(x_1, 2r)} u = 0$ . Hence by Theorem 4.6 we have

$$\text{ess sup}_{B(x_1, r)} u - \text{ess inf}_{B(x_1, r)} u \leq (1 - c_1^{-1}) \text{ess sup}_{B(x_1, r)} u \leq (1 - c_1^{-1}).$$

So if  $\rho = 1 - c_1^{-1}$  then

$$\text{ess sup}_{B(x_1, r)} u - \text{ess inf}_{B(x_1, r)} u \leq \rho [\text{ess sup}_{B(x_1, 2r)} u - \text{ess inf}_{B(x_1, 2r)} u].$$

It follows easily that

$$\text{ess sup}_{B(x_1, r)} u - \text{ess inf}_{B(x_1, r)} u \leq c_2 r^\gamma \tag{4.7}$$

for some  $\gamma > 0$ . Define  $\widehat{u}(x_1) = \lim_{r \rightarrow 0} \text{ess sup}_{B(x_1, r)} u$ . If one takes a countable basis  $\{B_i\}$  for  $X$  and excludes those points  $x \in B_i$  such that  $u(x) \notin [\text{ess inf}_{B_i} u, \text{ess sup}_{B_i} u]$ , then for every other  $x$  it is easy to see, using (4.7), that  $u(x) = \widehat{u}(x)$ . Thus,  $\widehat{u}$  is equal to  $u$  for  $\mu$ -almost every  $x$ . Moreover, from (4.7) we see that  $\widehat{u}$  is Hölder continuous. Recall that in our definition of harmonic function we take a quasi-continuous modification as defined in [FOT]. We conclude  $u = \widehat{u}$  quasi-everywhere, and so  $u$  has a quasi-continuous modification that is continuous. Using this modification and Theorem 4.6, we have

$$\sup_{B(x_0, R/2)} u \leq c_1 \inf_{B(x_0, R/2)} u.$$

The elliptic Harnack inequality EHI now follows by a covering argument.  $\square$

## 5. Stability under rough isometries.

In this section we prove Theorem 2.21. It is known (and easy) that the condition VD is stable under rough isometry, see [CS, HK2] etc.

In [HK2] the stability of  $\text{VD} + \text{PI}(\Psi)$  and  $\text{VD} + \text{CS}(\Psi)$  under rough isometries is proved in the case when the spaces are graphs with controlled weights. Since rough isometry is an equivalence relation, to prove Theorem 2.21 it is enough to prove that if  $X$  is a MMD space satisfying  $\text{VD}_{\text{loc}}$ ,  $\text{PI}(\beta)_{\text{loc}}$  and  $\text{CS}(\beta)_{\text{loc}}$  and  $G$  is a graph constructed by taking an appropriate net of  $X$ , (so that  $X$  and  $G$  are roughly isometric), then  $\text{VD}$ ,  $\text{PI}(\Psi)$  and  $\text{CS}(\Psi)$  hold for  $X$  if and only if they hold for  $G$ .

Let  $X$  be a MM space satisfying  $\text{VD}_{\text{loc}}$ . Let  $G \subset X$  be a maximal set such that

$$d(x, y) \geq 1 \text{ for } x, y \in G, x \neq y.$$

Thus  $B(x, \frac{1}{2})$ ,  $x \in G$  are disjoint, and  $\cup_{x \in G} B(x, 1) = X$ . Give  $G$  a graph structure by letting  $x \sim y$  if  $d(x, y) \leq 3$ . Let  $d_G$  be the usual graph distance on  $G$ , and write  $B_G(x, r)$  for balls in  $G$ . It is straightforward to check that  $G$  is connected, and that

$$\frac{1}{3}d(x, y) \leq d_G(x, y) \leq d(x, y) + 1, \quad x, y \in G.$$

Since  $X$  satisfies  $\text{VD}_{\text{loc}}$  we have, as in Lemma 2.3 of [Kan1], that the vertex degree in  $G$  is uniformly bounded.

For each  $x \sim y$  in  $G$  let  $z_{xy}$  be the midpoint of a geodesic connecting  $x$  and  $y$ , and  $A_{xy} = B(z_{xy}, 5/2)$ , so that  $B(x, 1) \subset A_{xy} \subset B(x, 4)$ . Let  $\nu_{xy} = 0$  if  $x \not\sim y$ , and if  $x \sim y$  let

$$\nu_{xy} = \mu(A_{xy}).$$

As usual we set  $\nu_x = \sum_{y \sim x} \nu_{xy}$ . Write  $A_x = \cup_{y \sim x} A_{xy}$ . Since  $X$  satisfies  $\text{VD}_{\text{loc}}$ , we have

$$\mu(B(x, 1)) \leq \nu_x \leq c_1 \mu(A_x) \leq c_1 \mu(B(x, 4)) \leq c_2 \mu(B(x, 1)), \quad (5.1)$$

and using (2.3) it is easy to verify that  $(G, \nu)$  has controlled weights.

Define  $\iota : G \rightarrow X$  by  $\iota(x) = x$ . We have

**Proposition 5.1.** *Let  $X$  be a MM space satisfying  $VD_{\text{loc}}$ . Then the associated weighted graph  $(G, \nu)$  has controlled weights and  $\iota$  is a rough isometry.*

In the following, we abuse notation and denote the image of  $\iota$  by the same character as its pre-image.

To prove Theorem 2.21(b) we will need to transfer functions between  $C(G, \mathbb{R}_+)$  and  $C(X, \mathbb{R}_+)$ . Let  $f \in C(X, \mathbb{R}_+)$ . Define

$$\widehat{f}(x) = \mu(B(x, 1))^{-1} \int_{B(x, 1)} f d\mu, \quad x \in G. \quad (5.2)$$

The transfer in the other direction requires a bit more care. Using  $VD_{\text{loc}}$  and  $CC_{\text{loc}}$ , we can find a partition of unity  $\{\psi_x\}_{x \in G}$  where each  $\psi_x : X \rightarrow \mathbb{R}$  is quasi continuous and satisfies the following:

- (i)  $\psi_x(w) = 1$  for  $w \in B(x, \frac{1}{4})$ ,
- (ii)  $\psi_x(w) = 0$  for  $w \in B(x, \frac{3}{2})^c$ ,
- (iii)  $\int_{B(x, 1)} d\Gamma(\psi_x, \psi_x) \leq cV(x, 1)$ .

If we assume  $CS(\bar{\beta})_{\text{loc}}$ , then we can choose  $\{\psi_x\}_{x \in G}$  that further satisfies the following:

- (iii') For each  $z \in G$ ,  $s \leq 1$  and  $f \in \mathcal{F}$ ,

$$\int_{B(z, s)} f^2 d\Gamma(\psi_x, \psi_x) \leq c_1 s^{2\theta} \left( \int_{B(z, 2s)} d\Gamma(f, f) + \Psi(s)^{-1} \int_{B(z, 2s)} f^2 d\mu \right). \quad (5.3)$$

Now if  $g : G \rightarrow \mathbb{R}_+$  set

$$\tilde{g}(z) = \sum_{x \in G} g(x) \psi_x(z). \quad (5.4)$$

Note that  $\tilde{g} : X \rightarrow \mathbb{R}$  is quasi continuous. Set also, if  $f : G \rightarrow \mathbb{R}$ ,  $k \in \mathbb{N}$ ,

$$V_k f(x) = \sup_{z: d_G(z, x) \leq k} |f(x) - f(z)|.$$

**Lemma 5.2.** *Let  $X$  be a MMD space satisfying  $VD_{\text{loc}}$  and  $CC_{\text{loc}}$ . Let  $f : G \rightarrow \mathbb{R}_+$ , and  $x \in G$ .*

- (a) *If  $\psi_z(w) > 0$  for some  $w \in B(x, 1)$  then  $d(x, z) < 3$  and  $x \sim z$ .*
- (b) *If  $\psi_z(w) > 0$  for some  $w \in A_x$  then  $d_G(x, z) \leq 4$ , and*

$$|f(x) - \tilde{f}(w)| \leq V_4 f(x), \quad w \in A_x.$$

- (c) *Let  $A \subset G$ , and  $A' = \{y : d_G(y, A) \leq 4\}$ . Then*

$$\sum_{z \in A} V_4 f(z)^2 \nu_z \leq c_1 \sum_{y, z \in A'} (f(y) - f(z))^2 \nu_{yz}.$$

(d)

$$c_2 \sum_{y \in G \cap B(x, r-1)} f(y)^2 \nu_y \leq \int_{B(x, r)} \tilde{f}(w)^2 d\mu(w) \leq c_3 \sum_{y \in G \cap B(x, r+2)} f(y)^2 \nu_y.$$

(e)

$$\int_{B(x, 1)} d\Gamma(\tilde{f}, \tilde{f}) \leq c_4 V_1 f(x)^2 V(x, 1).$$

(f) If  $X$  further satisfies  $CS(\bar{\beta})_{\text{loc}}$ . Then, for each  $s \leq 1$  and  $h \in \mathcal{F}$ ,

$$\int_{B(x, s)} h^2 d\Gamma(\tilde{f}, \tilde{f}) \leq c_5 s^{2\theta} V_1 f(x)^2 \left( \int_{B(x, 2s)} d\Gamma(h, h) + \Psi(s)^{-1} \int_{B(x, 2s)} h^2 d\mu \right).$$

*Proof.* The proof of (a)-(c) and the second inequality of (d) is simple, and is the same as that of [B2] Lemma 5.6 (a)-(d). So we will give the rest of the proof.

For the first inequality of (d), there is nothing to prove when  $r < 1$ , so assume  $r \geq 1$ . Using (5.1) and  $VD_{\text{loc}}$ , we see that the left hand side of (d) is bounded by

$$c \sum_{y \in G \cap B(x, r-1)} f(y)^2 V(y, 1/4) = c \sum_{y \in G \cap B(x, r-1)} \int_{B(y, 1/4)} \tilde{f}(w)^2 d\mu(w),$$

where the equality holds because for each  $y \in G$  and  $w \in B(y, 1/4)$ ,  $\tilde{f}(w) = f(y)$ . Since  $\{B(y, 1/4)\}_{y \in G}$  are disjoint, the right hand side is bounded from above by  $\int_{B(x, r)} \tilde{f}(w)^2 d\mu(w)$ .

Before proving (e), let us prove (f). By (a) we can write, for  $w \in B(x, 1)$ ,

$$\tilde{f}(w) = \tilde{f}(x) + \sum_{z \sim x} \psi_z(w)(f(z) - f(x)).$$

Hence

$$\begin{aligned} \int_{B(x, s)} h^2 d\Gamma(\tilde{f}, \tilde{f}) &= \sum_{z \sim x} \sum_{z' \sim x} (f(z) - f(x))(f(z') - f(x)) \int_{B(x, s)} h^2 d\Gamma(\psi_z, \psi_{z'}) \\ &\leq c \sum_{z \sim x} (f(z) - f(x))^2 \int_{B(x, s)} h^2 d\Gamma(\psi_z, \psi_z) \\ &\leq c \sum_{z \sim x} V_1 f(x)^2 \int_{B(x, s)} h^2 d\Gamma(\psi_z, \psi_z) \\ &\leq c' s^{2\theta} V_1 f(x)^2 \left( \int_{B(x, 2s)} d\Gamma(h, h) + \Psi(s)^{-1} \int_{B(x, 2s)} h^2 d\mu \right), \end{aligned} \tag{5.5}$$

where we used (5.3) in the last inequality.

Now (e) is easy. Using (5.5) with  $s = 1$  and  $h|_{B(x,2s)} \equiv 1$ , we have

$$\int_{B(x,1)} d\Gamma(\tilde{f}, \tilde{f}) \leq c \sum_{z \sim x} V_1 f(x)^2 \int_{B(x,1)} d\Gamma(\psi_z, \psi_z) \leq c' V_1 f(x)^2 V(x,1),$$

where  $CC_{\text{loc}}$  and  $VD_{\text{loc}}$  are used for the second inequality.  $\square$

**Lemma 5.3.** *Let  $X$  be a MMD space satisfying  $VD_{\text{loc}}$  and  $PI_{\text{loc}}$ . Let  $g : X \rightarrow \mathbb{R}_+$ ,  $x \in G$ , and  $y \sim x$ . Then*

$$(\hat{g}(x) - \hat{g}(y))^2 \nu_{xy} \leq c \int_{A_{xy}} d\Gamma(g, g).$$

The proof is simple, and it is the same as that of [B2] Lemma 5.7, so we omit it.

In the arguments that follow, we will use the fact, given in Remarks 2.6, that to verify  $CS(\Psi)$  it is enough to do so for any  $\delta > 0$  in (2.6) and  $\lambda > 0$  in (2.7).

**Proposition 5.4.** *Let  $X$  be a MMD space satisfying  $VD_{\text{loc}}$ ,  $PI_{\text{loc}}$  and  $CC_{\text{loc}}$ , and  $(G, \nu)$  be the associated weighted graph.*

- (a) *Suppose that  $X$  satisfies  $VD$  and  $PI(\Psi)$ . Then  $(G, \nu)$  satisfies  $VD$  and  $PI(\Psi)$ .*
- (b) *Suppose that  $X$  satisfies  $VD$  and  $CS(\Psi)$ . Then  $(G, \nu)$  satisfies  $VD$  and  $CS(\Psi)$ .*

*Proof.* As mentioned above,  $VD$  is preserved, so it is enough to prove  $PI(\Psi)$  and  $CS(\Psi)$  respectively.

We first prove (a). Let  $x \in G$ ,  $R > 0$ ,  $f : G \rightarrow \mathbb{R}$  and let  $B_G(x, R)$  be a ball in  $G$ . It is enough to prove the following weak Poincaré inequality;

$$\sum_{y \in B_G(x, R)} (f(y) - \bar{f}_{B_G(x, R)})^2 \nu_x \leq c \Psi(R) \sum_{y \in B_G(x, c'R)} \Gamma_G(f, f)(y), \quad (5.6)$$

where  $c > 0, c' > 1$  are constants. Indeed, an argument such as that in Jerison [Je] (see also [HaKo] for a more general formulation) can be used to derive the (strong) Poincaré inequality  $PI(\Psi)$  from  $VD$  and (5.6).

We note that for fixed  $c_0 \geq 1$ ,  $PI(\Psi)$  for  $R \leq c_0$  always holds on  $(G, \nu)$ . Indeed, if we let  $M_x := \max\{|f(x') - f(y')| : x', y' \in B_G(x_0, R)\}$ , then, since  $\min_{x \in B} f(x) \leq \bar{f}_B \leq \max_{x \in B} f(x)$ , we have that the left hand side of (5.6) is bounded by

$$2M_x \sum_{y \in B_G(x, R)} \nu_y \leq C_{c_0} \sum_{y \in B_G(x, R)} \Gamma_G(f, f)(y),$$

where  $C_{c_0} > 0$  depends on  $c_0$ . This in turn is less than or equal to the right hand side of (5.6). So it is enough to prove (5.6) for  $R \geq c_0$ . Applying Lemma 5.2.(d) for  $f - \alpha$  where  $\alpha = V(x, R)^{-1} \int_{B(x, R)} \tilde{f} d\mu$  and noting  $B_G(x, R') \subset G \cap B(x, R - 1)$  where  $R' = (R - 1)/3$ , we have

$$\int_{B(x, R)} (\tilde{f}(w) - \alpha)^2 d\mu(w) \geq c \sum_{y \in B_G(x, R')} (f(y) - \alpha)^2 \nu_y \geq c \sum_{y \in B_G(x, R')} (f(y) - \bar{f}_{B_G(x, R')})^2 \nu_y.$$

Here the last inequality is because the minimum of the middle term (as a function of  $\alpha$ ) is attained when  $\alpha = \bar{f}_{B_G(x,R')}$ . On the other hand, Using Lemma 5.2.(e), we have

$$\begin{aligned} \int_{B(x,R)} d\Gamma(\tilde{f}, \tilde{f}) &\leq \sum_{y \in B_G(x,R)} \int_{B(y,1)} d\Gamma(\tilde{f}, \tilde{f}) \leq c \sum_y V_1 f(y)^2 V(y,1) \\ &\leq c' \sum_{y \in B_G(x,R+2)} \Gamma_G(f, f)(y). \end{aligned}$$

Combining the estimates with PI( $\Psi$ ) for  $X$  gives (5.6) for  $R \geq c_0$ .

We next prove (b). We need to construct a cut-off function  $\hat{\varphi}$  satisfying (a)–(d) of Definition 2.5. If  $R \leq c_0$  then it is easy to check that we can take  $\hat{\varphi}(x)$  to be the indicator of  $B_G(x_0, R/2)$ .

So assume  $R > c_0$ . We can find a constant  $c_1$  such that

$$B_G(x_0, c_1 R) \subset G \cap B(x_0, R/8 - 6) \subset G \cap B(x_0, R/4 + 6) \subset B_G(x_0, R).$$

It is enough to construct a cut-off function  $\hat{\varphi}$  for  $B_G(x_0, c_1 R) \subset B_G(x_0, R)$ . Let  $\varphi$  be a cut-off function for  $B(x_0, R/8) \subset B(x_0, R/4)$ , and let  $\hat{\varphi}$  be given by (5.2). Properties (a)–(c) of Definition 2.5 are easily checked, and it remains to verify the weighted Sobolev inequality (2.5).

Let  $x_1 \in G$ ,  $1 \leq s \leq R$ , and  $A_G = B_G(x_1, s)$ . Choose  $c_2, c_3$  so that

$$A_G \subset B(x_1, c_2 s - 6) \cap G \subset B(x_1, 2c_2 s) \cap G \subset B_G(x_1, c_3 s - 6).$$

Write  $A'_G = B_G(x_1, c_3 s)$ , and let  $f : A'_G \rightarrow \mathbb{R}_+$ . We extend  $f$  to  $G$  by taking  $f$  to be zero outside  $A'_G$ , and define  $\tilde{f}$  by (5.4).

Let  $x \in G$ , and  $y \sim x$ . Then by Lemma 5.2(b) and Lemma 5.3

$$\begin{aligned} f(x)^2 (\hat{\varphi}(x) - \hat{\varphi}(y))^2 \nu_{xy} &\leq c \int_{A_{xy}} f(x)^2 d\Gamma(\varphi, \varphi)(w) \\ &\leq 2c \int_{A_{xy}} \tilde{f}(w)^2 d\Gamma(\varphi, \varphi)(w) + 2c \int_{A_{xy}} V_4 f(x)^2 d\Gamma(\varphi, \varphi)(w). \end{aligned}$$

Therefore

$$\begin{aligned} &\sum_{x \in A_G} \sum_{y \sim x} f(x)^2 (\hat{\varphi}(x) - \hat{\varphi}(y))^2 \nu_{xy} \\ &\leq c \sum_{x \in A_G} \sum_{y \sim x} \int_{A_{xy}} \tilde{f}(w)^2 d\Gamma(\varphi, \varphi)(w) + c \sum_{x \in A_G} \sum_{y \sim x} \int_{A_{xy}} V_4 f(x)^2 d\Gamma(\varphi, \varphi)(w) \\ &\leq c \int_{B(x_1, c_2 s)} \tilde{f}(w)^2 d\Gamma(\varphi, \varphi)(w) + c \sum_{x \in A_G} \sum_{y \sim x} V_4 f(x)^2 \int_{A_{xy}} d\Gamma(\varphi, \varphi)(w). \quad (5.7) \end{aligned}$$

Applying  $\text{CS}(\Psi)$  to  $\varphi$  in the ball  $A_{xy}$  gives

$$\int_{A_{xy}} d\Gamma(\varphi, \varphi) \leq cR^{-2\theta} \mu(B(z_{xy}, 5)) \leq c'R^{-2\theta} \nu_{xy}. \quad (5.8)$$

Therefore, using Lemma 5.2(c), the second term in (5.7) is bounded by

$$cR^{-2\theta} \sum_{x \in A_G} \sum_{y \sim x} V_4 f(x)^2 \nu_{xy} \leq cR^{-2\theta} \sum_{x \in A'_G} \Gamma_G(f, f)(x). \quad (5.9)$$

Using (2.5) for  $\varphi$  gives

$$\begin{aligned} & \int_{B(x_1, c_2 s)} \tilde{f}(w)^2 d\Gamma(\varphi, \varphi)(w) \\ & \leq c(s/R)^{2\theta} \left( \int_{B(x_1, 2c_2 s)} d\Gamma(\tilde{f}, \tilde{f}) + \Psi(s)^{-1} \int_{B(x_1, 2c_2 s)} \tilde{f}^2 d\mu \right). \end{aligned} \quad (5.10)$$

By Lemma 5.2(e),

$$\begin{aligned} \int_{B(x_1, 2c_2 s)} d\Gamma(\tilde{f}, \tilde{f}) & \leq \sum_{x \in G \cap B(x_1, 2c_2 s + 1)} \int_{B(x, 1)} d\Gamma(\tilde{f}, \tilde{f}) \leq c \sum_{x \in G \cap B(x_1, 2c_2 s + 1)} V_1 f(x)^2 V(x, 1) \\ & \leq c' \sum_{x, y \in A'_G} (f(x) - f(y))^2 \nu_{xy}, \end{aligned} \quad (5.11)$$

while by Lemma 5.2(d)

$$\int_{B(x_1, 2c_2 s)} \tilde{f}(w)^2 d\mu(w) \leq \sum_{x \in B_G(x_1, c_3 s)} f(x)^2 \nu_x. \quad (5.12)$$

Combining the estimates (5.7)–(5.12) completes the proof.  $\square$

**Proposition 5.5.** *Let  $X$  be a MMD space satisfying  $\text{VD}_{\text{loc}}$ ,  $\text{PI}_{\text{loc}}$ , and  $(G, \nu)$  be the associated weighted graph.*

(a) *Suppose that  $X$  satisfies  $\text{PI}(\bar{\beta})_{\text{loc}}$ . If  $(G, \nu)$  satisfies  $\text{VD}$  and  $\text{PI}(\Psi)$  then  $X$  satisfies  $\text{VD}$  and  $\text{PI}(\Psi)$ .*

(b) *Suppose that  $X$  satisfies  $\text{CS}(\bar{\beta})_{\text{loc}}$ . If  $(G, \nu)$  satisfies  $\text{VD}$  and  $\text{CS}(\Psi)$  then  $X$  satisfies  $\text{VD}$  and  $\text{CS}(\Psi)$ .*

*Proof.* We first prove (a). Let  $x_0 \in X$ ,  $R > 0$  and  $f \in \mathcal{F}$ . As mentioned in the proof of Proposition 5.4.(a), it is enough to prove the following weak Poincaré inequality;

$$\int_{B(x_0, R)} (f(y) - \bar{f}_{B(x_0, R)})^2 \mu(y) \leq c\Psi(R) \int_{B(x_0, c'R)} d\Gamma(f, f), \quad (5.13)$$

where  $c > 0, c' > 1$  are constants. When  $R \leq 1$ , this can be obtained from  $\text{PI}(\bar{\beta})_{\text{loc}}$ , so assume  $R > 1$ . Using  $\text{PI}(\bar{\beta})_{\text{loc}}$  with  $R = 1$ , we have for each  $x \in G$ ,

$$\int_{B(x,1)} f^2 d\mu - \widehat{f}(x)^2 \nu_x = \int_{B(x,1)} (f - \widehat{f})^2 d\mu \leq c \int_{B(x,1)} d\Gamma(f, f).$$

Summing this over  $x \in B(x_0, R) \cap G$ , we have

$$\int_{B(x_0, R)} f^2 d\mu \leq c \left( \sum_{x \in B(x_0, R) \cap G} \widehat{f}(x)^2 \nu_x + \int_{B(x_0, R+1)} d\Gamma(f, f) \right).$$

Putting  $f - \alpha$  instead of  $f$  where  $\alpha = \sum_{x \in B(x_0, R) \cap G} \widehat{f}(x) \nu_x / (\sum_{x \in B(x_0, R) \cap G} \nu_x)$  and using the fact that  $\int_{B(x_0, R)} (f - \beta)^2 d\mu$  (as a function of  $\beta$ ) attains its minimum when  $\beta = \bar{f}_{B(x_0, R)}$ , we obtain

$$\int_{B(x_0, R)} (f - \bar{f}_{B(x_0, R)})^2 d\mu \leq c \left( \sum_{x \in B(x_0, R+1) \cap G} (\widehat{f}(x) - \alpha)^2 \nu_x + \int_{B(x_0, R+1)} d\Gamma(f, f) \right).$$

On the other hand, using Lemma 5.3. and summing, we have

$$\sum_{x \in B(x_0, R+1) \cap G} \Gamma_G(\widehat{f}, \widehat{f})(x) \leq c \int_{B(x_0, R+5)} d\Gamma(f, f).$$

Combining the estimates with  $\text{PI}(\Psi)$  for  $G$  gives (5.13) for  $R \geq 1$ .

We next prove (b). Let  $B = B(x_0, R)$  be a ball in  $X$ . If  $R \leq c_1$  then we can use  $\text{CS}(\bar{\beta})_{\text{loc}}$  to construct a cut-off function  $\varphi$  for  $B$ . So assume  $R \geq c_1$ . We can therefore assume that  $x_0 \in G$ .

Given  $A \subset G$  write  $A^{(1)} = \cup_{x \in A} B(x, 1)$ . We can find  $c_i$  such that

$$B(x_0, c_1 R) \subset B_G(x_0, c_2 R - 6)^{(1)} \subset B_G(x_0, 2c_2 R + 6)^{(1)} \subset B(x_0, R).$$

Let  $\varphi_G$  be a cut-off function for  $B_G(x_0, c_2 R) \subset B_G(x_0, 2c_2 R)$ , and let

$$\varphi(w) = \widetilde{\varphi}_G(w) = \sum_{z \in G} \varphi_G(x) \psi_z(w).$$

Properties (a)–(c) of  $\varphi$  follow easily from those of  $\varphi_G$ , and it remains to verify (2.5).

Let  $B_1 = B(x_1, s)$  with  $s \in (0, R)$ . If  $s \leq c_3$  then, applying Lemma 5.2(f) and noting  $V_1 \varphi(x) \leq cR^{-2\theta}$ , we have

$$\int_{B(x_1, s)} g^2 d\Gamma(\varphi, \varphi) \leq c(s/R)^{2\theta} \left( \int_{B(x_1, 2s)} d\Gamma(g, g) + \Psi(s)^{-1} \int_{B(x_1, 2s)} g^2 d\mu \right).$$

Now suppose  $s \geq c_3$ . Then we can assume  $x_1 \in G$ , and there exist  $c_i$  so that

$$B(x_1, s) \subset B_G(x_1, c_4 s - 6)^{(1)} \subset B_G(x_1, 2c_4 s + 6)^{(1)} \subset B(x_1, c_5 s - 6).$$

Let  $g : B(x_1, c_5 s) \rightarrow \mathbb{R}_+$ . Define  $\hat{g}$  on  $B_G(x_1, 2c_4 s + 6)$  by (5.2). Then

$$\begin{aligned} \int_{B(x_1, s)} g^2 d\Gamma(\varphi, \varphi) &\leq \sum_{x \in B_G(x_1, c_4 s)} \int_{B(x, 1)} g(w)^2 d\Gamma(\varphi, \varphi)(w) \\ &\leq 2 \sum_{x \in B_G(x_1, c_4 s)} \int_{B(x, 1)} (g(w) - \hat{g}(x))^2 d\Gamma(\varphi, \varphi)(w) \end{aligned} \quad (5.14)$$

$$+ 2 \sum_{x \in B_G(x_1, c_4 s)} \int_{B(x, 1)} \hat{g}(x)^2 d\Gamma(\varphi, \varphi)(w). \quad (5.15)$$

By Lemma 5.2(e) with  $s = 1$ , the term (5.14) is bounded by

$$cR^{-2\theta} \sum_{x \in B_G(x_1, c_4 s)} \left( \int_{B(x, 1)} (g(w) - \hat{g}(x))^2 d\mu + \int_{B(x, 1)} d\Gamma(g, g) \right),$$

and using  $\text{PI}_{\text{loc}}$  this is bounded by

$$c'R^{-2\theta} \sum_{x \in B_G(x_1, c_4 s)} \int_{B(x, 1)} d\Gamma(g, g) \leq c''R^{-2\theta} \int_{B(x_1, c_5 s)} d\Gamma(g, g). \quad (5.16)$$

For the term (5.15), by Lemma 5.2(e) and (2.11) for  $\varphi_G$ ,

$$\begin{aligned} &\sum_{x \in B_G(x_1, c_4 s)} \hat{g}(x)^2 \int_{B(x, 1)} d\Gamma(\varphi, \varphi)(w) \\ &\leq \sum_{x \in B_G(x_1, c_4 s)} \hat{g}(x)^2 V_1 \varphi_G(x)^2 V(x, 1) \\ &\leq c \sum_{x \in B_G(x_1, c_4 s)} \hat{g}(x)^2 \Gamma_G(\varphi_G, \varphi_G)(x) \\ &\leq c(s/R)^{2\theta} \left( \sum_{x \in B_G(x_1, 2c_4 s)} \Gamma_G(\hat{g}, \hat{g})(x) + \Psi(s)^{-1} \sum_{x \in B_G(x_1, 2c_4 s)} \hat{g}(x)^2 \nu_x \right). \end{aligned}$$

Using Lemma 5.3 for the first term, and an easy bound for the second, (2.5) now follows.  $\square$

We finally mention several applications of our results.

### 1. Dirichlet forms on generalized Sierpinski carpets

In [BB1, BB2] a ‘nice’ diffusion process on a generalized Sierpinski carpet  $F$  is constructed and it is proved that its heat kernel satisfies  $\text{HK}(\Psi)$  with  $\Psi(s) = s^{d_w}$  for some  $d_w \geq 2$ . Let the corresponding Dirichlet form be denoted by  $\mathcal{E}_1$ . On the other hand, a self-similar Dirichlet form  $\mathcal{E}_2$  was constructed on the carpet by an averaging method in [KZ] (see also [BB2] Remark 5.11 and [HKKZ]). Because of the possible lack of uniqueness for the ‘nice’ diffusion, it is not known if the corresponding diffusions coincide. Further, it was not clear that the heat kernel corresponding to  $\mathcal{E}_2$  satisfies  $\text{HK}(\Psi)$ . (If one tries to apply the coupling methods used in [BB2], this works for a dense set of starting points, but not for all starting points because one does not know *a priori* the continuity of harmonic functions.)

**Theorem 5.6.** *Let  $(F, d)$  be a generalized Sierpinski carpet with walk dimension  $d_w$ , and  $\mathcal{E}_2$  be the Dirichlet form constructed in [KZ]. Then the MMD space  $(F, d, \mathcal{E}_2)$  satisfies  $\text{HK}(d_w)$  and  $\text{PHI}(d_w)$ .*

*Proof.* It is easy to check that  $\mathcal{E}_2$  is a bounded perturbation of  $\mathcal{E}_1$  in the sense of (2.12) (see [Hin] Section 5.2 and [KZ] Section 6). Thus  $\text{HK}(\Psi)$  for  $\mathcal{E}_2$  holds by Theorem 2.19.  $\square$

In [Hin], general criteria are given for energy measures of self-similar Dirichlet forms on self-similar sets to be singular with respect to Bernoulli type measures. Roughly speaking, the main theorems imply that the energy measures are singular with respect to the Hausdorff measure if the elliptic Harnack inequality holds and the walk dimension of the corresponding process (which corresponds to  $\beta$  and  $\bar{\beta}$  in this paper) is greater than 2. As a consequence of the last paragraph, these conditions hold for the self-similar Dirichlet forms on the Sierpinski carpets and we can conclude that the energy measures of the Dirichlet forms (both for  $\mathcal{E}_1$  and  $\mathcal{E}_2$  above) are singular with respect to Hausdorff measure. (In fact, we can also verify (O) in [Hin], so using Theorem 2.4 in [Hin] we can further conclude the singularity with respect to a certain class of Bernoulli type measures.) Note that this was not proved in [Hin] for higher dimensional carpets since at that point the elliptic Harnack inequality had not been proved for the self-similar Dirichlet form  $\mathcal{E}_2$ .

## 2. Weighted graphs and manifolds associated with MM spaces satisfying $\text{VD}_{\text{loc}}$

In this section, we have explained how to construct weighted graphs from MM spaces satisfying  $\text{VD}_{\text{loc}}$ . There is a natural way to construct ‘jungle gym’ type manifolds from the weighted graphs (see [BCG, Kan2, PS] etc.). Roughly, this can be done by replacing the edges by tubes of length 1, and by gluing the tubes together smoothly at the vertices (see the right side of Figure 1). By Theorem 2.21, we know that if the original MM space satisfies  $\text{PHI}(\Psi)$  for some  $\Psi(s) = s^{\bar{\beta}} \vee s^{\beta}$ , then the network on the associated weighted graph and the Laplacian on the associated manifold satisfy  $\text{PHI}(\Psi')$  with  $\Psi'(s) = s^2 \vee s^{\beta}$ . Further, any uniformly elliptic operator in divergence form on a manifold which is roughly isometric to the MMD space satisfies  $\text{PHI}(\Psi')$ .

This fact is useful in fractal contexts, since  $\text{PHI}(\Psi)$  with  $\Psi(s) = s^{d_w}$  for some  $d_w \geq 2$  is proved for various ‘regular’ fractals. Our result thus gives an alternative proof of the results in [BB3, BB4, Jo] and the heat kernel results in [HK2].

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