The Liouville property and a conjecture of de Giorgi

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\textbf{Abstract.} We consider bounded entire solutions of the non-linear PDE $\Delta u + u - u^3 = 0$ in $\mathbb{R}^d$, and prove that under certain monotonicity conditions these solutions must be constant on hyperplanes. The proof uses a Liouville theorem for harmonic functions associated with a non-uniformly elliptic divergence form operator.

0. Introduction.

In 1978 De Giorgi [Gi] formulated the following:

\textbf{Conjecture.} Suppose that $u$ is an entire solution of the equation

$$\Delta u + u - u^3 = 0$$

satisfying

$$|u(x)| \leq 1, \quad \frac{\partial u}{\partial x_d}(x) > 0 \quad \text{for} \quad x = (x', x_d) \in \mathbb{R}^d,$$

$$\lim_{x_d \to -\infty} u(x', x_d) = 1, \quad \text{and} \quad \lim_{x_d \to -\infty} u(x', x_d) = -1.$$ 

Then the level sets of $u$ must be hyperplanes, i.e. there exists $g \in C^2(\mathbb{R})$ such that $u(x) = g(a \cdot x)$, for some fixed $a \in \mathbb{R}^d$ with $|a| = 1$.

Let $F \in C^{2+\epsilon}(\mathbb{R})$ be a non-negative function such that $F(\pm 1) = 0$ and $F''(\pm 1) \geq \mu > 0$ for some constant $\mu$. A more general form of (0.1) is the equation

$$\Delta u - F'(u) = 0, \quad \text{for} \quad x = (x', x_d) \in \mathbb{R}^d,$$

where

$$|u(x)| \leq 1, \quad \frac{\partial u}{\partial x_d}(x) > 0 \quad \text{for} \quad x = (x', x_d) \in \mathbb{R}^d,$$

$$\lim_{x_d \to -\infty} u(x', x_d) = 1, \quad \text{and} \quad \lim_{x_d \to -\infty} u(x', x_d) = -1.$$ 

A generalization of the De Giorgi conjecture is that any solution of (0.3)-(0.4) is constant on hyperplanes, and so of the form $u(x) = g(a \cdot x)$, for some fixed $a \in \mathbb{R}^d$ with $|a| = 1$. It is clear that the function $g$ must be a solution of the ODE

$$g''(t) - F'(g(t)) = 0, \quad t \in \mathbb{R}, \quad |g(t)| \leq 1, \quad \text{and} \quad \lim_{t \to \pm \infty} g(t) = \pm 1.$$ 

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This ODE has a solution which is unique up to translation. Note that in (0.1) we have 
\[ F(u) = \frac{1}{4}(u^2 - 1)^2, \] which satisfies the conditions above with \( F''(u) = 3u^2 - 1. \)

It is known [GT] that any bounded solution \( u \) of (0.3) is \( C^{3+\epsilon} \) in \( \mathbb{R}^d \). In [MM] and [CGS], it is shown that any bounded solution of (0.3) satisfies the gradient bound

\[ |\nabla u(x)|^2 \leq 2F(u(x)) \quad \text{for all } x \in \mathbb{R}^d. \quad (0.6) \]

It is also proved there that the (generalized) De Giorgi conjecture is true in any dimension for any solution \( u \) such that equality in (0.6) holds at some point \( x_0 \in \mathbb{R}^d \). Also, in [MM] it is proved that if \( d = 2 \) then the de Giogi conjecture holds for any solution \( u \) for which the level sets are the graphs of an equilipschitzian family of functions. See also [M1], [M2] and [CGS] for other results, and also [DFP] for the existence of some entire solutions of (0.1) of a quite different form. [GNN] obtained some striking results on a related problem.

Motivated by a problem in cosmology, G.W. Gibbons (see [C]) made the weaker conjecture that the level sets of \( u \) are hyperplanes if \( u \) satisfies (0.1), (0.2), and the additional condition that the convergence of \( u(x', x_3) \) to \( \pm 1 \) is uniform as \( x_3 \) tends \( \pm \infty \).

Recently, in [GG] Ghoussoub and Gui proved the De Giorgi conjecture for \( d = 2 \) without any extra assumptions, and proved Gibbons conjecture for \( d = 3 \).

Our first result is a proof of Gibbons’ conjecture for \( d \geq 3 \).

**Theorem 1.** Suppose that \( u(x) \) satisfies (0.3), converges to 1 uniformly as \( x_d \) tends to \( \infty \), and converges to \(-1\) uniformly as \( x_d \) tends to \(-\infty \). Then \( u \) is necessarily of the form \( u(x', x_d) = g(x_d) \), where \( g(t) \) is a solution of (0.5).

We can relax the uniform convergence condition if make some additional assumptions on \( F \), and assume that the level sets of \( u \) are Lipschitzian.

**Theorem 2.** Assume that \( F(u) \) in (0.3) has only one critical point \( u_0 \) in \((-1, 1)\) and that \( F''(u_0) < 0 \). Suppose that \( u(x) \) satisfies (0.3), \( u(x', x_d) \to 1 \) as \( x_d \to \infty \), \( u(x', x_d) \to -1 \) as \( x_d \to -\infty \), and that the level sets of \( u(x', x_d) \) are the graphs of Lipschitzian functions of \( x' \), i.e. there exists a continuous positive function \( L(b) \) for \( b \in (-1, 1) \) such that

\[ |\nabla_{x'} u(x)| \leq L(u(x)) \frac{\partial u(x)}{\partial x_d}, \quad x \in \mathbb{R}^d. \]

Then \( u \) is necessarily of the form \( u(x', x_d) = g(a \cdot x) \) for some \( a \in \mathbb{R}^d \) with \( |a| = 1 \), where \( g(t) \) is a solution of (0.5).

**Remarks.** 1. Note that \( F(u) = \frac{1}{4}(u^2 - 1)^2 \) satisfies the conditions of Theorem 2.

2. The Lipschitzian condition on \( u \) in Theorem 2 is weaker than that in [MM], where (as well as taking \( d = 2 \)) \( L(b) \) is assumed to be bounded. Indeed, all we need is that \( L(b) < \infty \) on an interval \([-1 + \delta, 1 - \delta]\), where the constant \( \delta > 0 \) depends only on \( F \).

3. Let \( e^{(d)} = (0, 1) \in \mathbb{R}^{d-1} \times \mathbb{R} \) be the unit vector in the \( x_d \) direction. It is easy to see that the Lipschitzian condition on \( u \) in the above theorem is equivalent to the following monotonicity condition of \( u \) in a small cone: for any \( b \in (-1, 1) \) there exists a \( \delta_0(b) > 0 \) such that if \( |\nu| = 1 \) then

\[ \nu \cdot \nabla u(x) > 0 \quad \text{whenever } \nu \cdot e^{(d)} > 1 - \delta_0(u(x)), \quad x \in \mathbb{R}^d. \]
We have recently learnt that Theorem 1 (but not Theorem 2) has also been proved, using different methods, in [BCM] and [F].

The proof of both Theorems 1 and 2 employs the same basic strategy, which uses ideas introduced by Ghoussoub and Gui in [GG]. Let

\[ \sigma(x) = \frac{\partial u(x)}{\partial x_d}. \]  (0.7)

In the case of Theorem 2, \( \sigma(x) > 0 \) in \( \mathbb{R}^d \) by hypothesis, while it is shown in [GG] by using the moving plane method that the hypotheses of Theorem 1 imply that \( \sigma(x) > 0 \) in \( \mathbb{R}^d \). For \( a \in \mathbb{R}^d \) with \( |a| = 1 \) define the directional derivative \( \psi_a(x) = a \cdot \nabla u(x) \). Differentiating (0.3) we have that both \( \sigma \) and \( \psi_a \) satisfy

\[ \Delta \varphi - F''(u(x)) \varphi = 0, \quad x \in \mathbb{R}^d. \]

Let

\[ h(x) = \frac{\psi_a(x)}{\sigma(x)}, \]

and set

\[ \mathcal{L} = \frac{1}{2} \sigma^{-2} \nabla (\sigma^2 \nabla) = \frac{1}{2} \Delta + (\sigma^{-1} \nabla \sigma) \nabla. \]  (0.8)

Then \( h \) is \( \mathcal{L} \)-harmonic, since

\[ 2\mathcal{L}h = \sigma^{-2} \nabla (\sigma^2 \nabla h) = \sigma^{-2} \nabla (\psi_a \nabla \sigma - \sigma \nabla \psi_a) = \sigma^{-2} (\psi_a \Delta \sigma - \sigma \Delta \psi_a) = 0. \]

Note also that \( \sigma h = \psi_a \) is bounded, by (0.6).

Suppose that the operator \( \mathcal{L} \) satisfies the Liouville property in the form:

(LP) \[ \text{If } h \text{ satisfies } \mathcal{L}h = 0 \text{ in } \mathbb{R}^d \text{ and } \sigma h \text{ is bounded, then } h \text{ is constant.} \]

Then for each \( a \) there exists a constant \( c(a) \) such that

\[ \psi_a(x) = a \cdot \nabla u(x) = c(a) \sigma(x), \quad x \in \mathbb{R}^d. \]  (0.9)

It follows immediately from (0.9) that \( u \) is constant on any hyperplane orthogonal to \( \nabla u(0) \).

Thus the proof of Theorems 1 and 2 reduces to establishing the Liouville property (LP). If \( \sigma \) is any \( C^2 \) function on \( \mathbb{R}^d \) satisfying \( \sigma \geq \varepsilon > 0 \), and \( \mathcal{L} = \mathcal{L}_\sigma \) is defined by (0.8), then (LP) is well known. However (LP) may fail for general \( \sigma > 0 \) – see [GG] and [Ba] for counterexamples in the cases \( d \geq 7, d \geq 3 \) respectively. The proof in [Ba] is probabilistic, and shows that the Liouville property fails for suitable \( \mathcal{L}(= \mathcal{L}_\sigma) \) by proving non-trivial tail behaviour of the diffusion process \( X = (X_t, t \geq 0) \) associated with \( \mathcal{L} \). However for \( \sigma \) arising from (0.7), the bound (0.6) implies that

\[ \sigma(x', x_d) \to 0 \text{ as } |x_d| \to \infty \text{ for each } x' \in \mathbb{R}^{d-1}. \]  (0.10)
As $X$ tends to avoid regions where $\sigma$ is small, (0.10) suggests that, in the case of Theorem 1, where the convergence is uniform, the process $X$ largely lives on some ‘slab’ $D$ of the form $D = \mathbb{R}^{d-1} \times [-c, c]$. Since (see Section 4) one can prove that $\sigma(x) > \varepsilon_1 > 0$ for $x \in D$, $X$ is in some sense close to a uniformly elliptic divergence form diffusion, which suggests that (LP) should hold for $X$ and $\mathcal{L}$.

Some additional smoothness conditions are needed to establish the Liouville property. In the theorem below, the simplest case, which is sufficient to prove Theorem 1, is when $\gamma = 0$.

**Theorem 3.** Let $\gamma : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ be $C^2$, with $|\nabla \gamma(x')| \leq K_0$, $x' \in \mathbb{R}^{d-1}$, for some constant $K_0 < \infty$. For $-\infty \leq a \leq b \leq \infty$ write

$$I(a, b) = \{(x', x_d) \in \mathbb{R}^d : \gamma(x') + a \leq x_d \leq \gamma(x') + b\}.$$

Let $\sigma : \mathbb{R}^d \rightarrow (0, \infty)$ be a $C^2$ function, and let $\mathcal{L} = \frac{1}{2}\sigma^{-2}\nabla(\sigma^2 \nabla)$. Suppose that there exist constants $0 < \varepsilon_0 < 1$, $1 \leq K_1 < K_2 < \infty$, $K_3 < \infty$ such that $\sigma$ satisfies

(S1) $\sigma^{-1}\Delta \sigma \geq 2\varepsilon_0$ on $I(-K_1, K_1)$,

(S2) $\sigma \geq \varepsilon_0/2$ on $I(-K_2, K_2)$,

(S3) $||\sigma||_\infty$, $||\nabla \sigma||_\infty$ and $||\Delta \sigma||_\infty$ are all bounded by $K_3$.

Then if $Lh = 0$ and $\sigma h$ is bounded, then $h$ is constant.

**Remark.** Though we will not use this fact, these conditions on $\sigma$ imply that $\sigma(x', x_d) \rightarrow 0$ as $|x_d| \rightarrow \infty$.

Write

$$H(\lambda) = I(\lambda, \lambda), \quad \lambda \in \mathbb{R}.$$

Let $X$ be the diffusion associated with $\mathcal{L}$. One approach to Liouville theorems such as Theorem 3 is to obtain global upper and lower bounds on the transition density $k(t, x, y)$ of $X$, which is the solution to the heat equation

$$\mathcal{L}k = \frac{\partial k}{\partial t}.$$

There is a substantial literature on bounds of this type, but with most approaches some kind of uniform ellipticity condition on $\mathcal{L}$ is essential. We avoid this difficulty by considering instead the time-change of the process $X$ on the submanifold $H(0)$. Write $\tilde{X}$ for this process, and let $Y$ be the projection of $\tilde{X}$ onto $\mathbb{R}^{d-1}$. $Y$ is a pure jump process with generator of the form

$$\mathcal{L}_Y f(x) = \int_{\mathbb{R}^{d-1}} (f(y') - f(x')) n(x', y') dy',$$

where $n$ is symmetric and continuous away from the diagonal. Let $q = q(t, x', y')$ be the transition density of $Y$: $q$ solves the equation

$$\mathcal{L}_Y q = \frac{\partial q}{\partial t}.$$
We obtain upper and lower bounds on \( q \), and from these prove a Liouville theorem for \( \mathcal{L}_Y \)-harmonic functions. Theorem 3 then follows easily.

The contents of this paper are as follows. In Section 1 we consider jump processes \( Y \) given by (0.11), and, under suitable conditions on the function \( n(x,y) \), which include exponential decay as \( |x-y| \to \infty \), we obtain upper and lower bounds on \( q \) and prove a Liouville theorem for \( \mathcal{L}_Y \). In Section 2 we use Girsanov’s transformation to construct the diffusion \( X \) associated with \( \mathcal{L} \). The main result in this section is an exponential bound on \( |X_{\tau_0} - X_0| \), where \( \tau_0 \) is the first hitting time of \( H(0) \). Section 3 deals with the construction of the processes \( \tilde{X} \) and \( Y \) from \( X \), and estimates on the jump measures \( n \). The exponential bounds on \( |X_{\tau_0} - X_0| \) lead to exponential decay of \( n(x,y) \) as \( |x-y| \to \infty \). Finally, in Section 4 we complete the proof of Theorems 1 and 2, by showing that the function \( \sigma = \partial u(x)/\partial x_d \) satisfies the conditions of Theorem 3.

We write \( c_i \) for unimportant positive finite constants; these are fixed within each lemma, proposition, theorem and corollary.

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1. Heat kernel of a jump process.

Let \( N \) be a measure on \( \mathbb{R}^n \times \mathbb{R}^n - D \) (where \( D \) is the diagonal) with a symmetric density \( n(x,y) \). Throughout this section we will assume that there exist constants \( \alpha_0 > 0, c_i \) such that

\[
\int_{|y-x| > r} n(x,y) dy \leq c_0 e^{-\alpha_0 r}, \quad r \geq 1, \tag{1.1}
\]

\[
c_1 |x-y|^{-(n+1)} \leq n(x,y) \leq c_2 |x-y|^{-(n+1)}, \quad |x-y| \leq 1. \tag{1.2}
\]

Let \( \mathcal{L}_Y \) be the generator

\[
\mathcal{L} f(x) = \int_{\mathbb{R}^n} (f(y) - f(x))n(x,y)dy, \quad f \in C^\infty_0(\mathbb{R}^n),
\]

and \( \mathcal{E} \) be the Dirichlet form on \( L^2(\mathbb{R}^n, dx) \) with core \( C^\infty_0(\mathbb{R}^n) \) given by

\[
\mathcal{E}(f,f) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} (f(y) - f(x))^2 n(x,y)dxdy, \quad f \in C^\infty_0(\mathbb{R}^n). \]

(An argument similar to that in [FOT, p. 100] implies that \( \mathcal{E} \) is regular). Let \( Y = (Y_t, t \geq 0, \mathbb{Q}_x, x \in \mathbb{R}^n) \) be the symmetric Markov process associated with \( \mathcal{E} \). Set also

\[
n_0(x,y) = |x-y|^{-(n+1)}1_{|x-y| < 1}, \quad x, y \in \mathbb{R}^n, x \neq y,
\]

Replacing \( n \) by \( n_0 \) in the equations above, let \( \mathcal{L}_0 \) and \( \mathcal{E}_0 \) be the corresponding generator and Dirichlet form of the Markov process \( Y^0 = (Y^0_t, t \geq 0, \mathbb{Q}^x_0, x \in \mathbb{R}^n) \).
From (1.2) we have
\[ \mathcal{E}(f, f) \geq c_1 \mathcal{E}_0(f, f), \quad f \in C_0^\infty(\mathbb{R}^n). \] (1.3)

The process \( Y^0_t \) is a Lévy process, and therefore \( Y^0_t \) has characteristic function \( \psi(\lambda) \), given by
\[ \mathbb{E}^0 e^{i \lambda \cdot Y^0_t} = e^{-t \psi(\lambda)}, \quad \lambda \in \mathbb{R}^n, \]
where, since \( Y^0 \) is symmetric,
\[ \psi(\lambda) = \int_{\mathbb{R}^n} \left(1 - \cos \lambda \cdot x\right) n_0(0, x) \, dx. \] (1.4)

**Lemma 1.1.** For each \( t > 0 \), under \( Q^0_0 \), \( Y^0_t \) has a continuous density \( q^0_t(x) \), \( x \in \mathbb{R} \), which satisfies
\[ q^0_t(x) \leq c_1 t^{-n/2}, \quad t \geq 1. \] (1.5)
\[ q^0_t(x) \leq c_1 t^{-n}, \quad t \leq 1. \] (1.6)

**Proof.** By the radial symmetry of (1.4) we have
\[ \psi(\lambda) = \int_{|x|<1} \left(1 - \cos(x_1|\lambda|)|x|^{-n-1} \, dx = |\lambda| \int_{|y|<|\lambda|} (1 - \cos y_1)|y|^{-n-1} \, dy. \]

Hence if \( |\lambda| \geq 1 \) then \( \psi(\lambda) \geq c_2 |\lambda| \), while if \( |\lambda| < 1 \) then \( 1 - \cos x_1|\lambda| \geq c_3 x_1^2|\lambda|^2 \), so
\[ \psi(\lambda) \geq c_3 |\lambda|^2 \int_{|x|<1} x_1^2 |x|^{-n-1} \, dx \geq c_4 |\lambda|^2. \]

Therefore \( \int |\lambda|^p e^{-t \psi(\lambda)} \lambda \, d\lambda < \infty \) for any \( p < \infty \), and by Fourier inversion \( Y^0_t \) has a \( C^\infty \) density \( q^0_t(\cdot) \).

Also, by the Fourier inversion formula,
\[ q_t(x) = \int_{\mathbb{R}^n} e^{-i \lambda \cdot x} e^{-t \psi(\lambda)} \, d\lambda \]
\[ \leq q_t(0) = \int_{\mathbb{R}^n} e^{-t \psi(\lambda)} \, d\lambda \]
\[ \leq \int_{|\lambda| \leq 1} e^{-c_4 t|\lambda|^2} \, d\lambda + \int_{|\lambda| > 1} e^{-c_2 t|\lambda|^2} \, d\lambda \]
\[ = c_5 \int_0^1 r^{n-1} e^{-c_4 t r^2} \, dr + c_5 \int_1^\infty r^{n-1} e^{-c_2 t r} \, dr. \]

Estimating these integrals, the bounds (1.5), (1.6) follow easily. \( \square \)

We can now use Theorem 2.9 of [CKS] to deduce similar estimates for \( Y \).
Theorem 1.2. \( Y \) has a transition density \( q_t(x, y) \) which satisfies

\[
q_t(x, y) \leq c_1 t^{-n/2}, \quad t \geq 1. \tag{1.7}
\]
\[
q_t(x, y) \leq c_1 t^{-n}, \quad t \leq 1. \tag{1.8}
\]

Proof. Write \( q_t^0(x, y) \) for the transition densities of \( Y^0 \). As \( Y^0 \) is a Lévy process \( q_t^0(x, y) = q_t^0(y - x) \). From Lemma 1.1 we have, for a suitable \( c_2 < 1 \),

\[
q_t^0(x, y) \leq c_2 t^{-n} e^t, \quad t > 0.
\]

So, by Theorem 2.1 of [CKS], and writing \( m = 2n \), \( E_0 \) satisfies a Nash inequality

\[
\| f \|_2^{2+4/m} \| f \|_1^{-4/m} \leq c_3 \left[ E_0(f, f) + \| f \|_2^2 \right].
\]

Using (1.3), \( E \) satisfies a Nash inequality of the same form, and hence, by the converse implication in [CKS, Theorem 2.1], \( Y \) has a transition density \( q_t(x, y) \) which satisfies

\[
q_t(x, y) \leq c_4 t^{-n} e^t, \quad t > 0.
\]

The bound (1.8) is immediate.

To obtain bounds for \( t \geq 1 \), we use the conditional Nash inequalities discussed in [CKS, Theorem 2.9]. First, from (1.5) it follows that \( E_0 \) also satisfies

\[
\| f \|_2^{2+4/n} \| f \|_1^{-4/n} \leq c_5 E_0(f, f) \quad \text{whenever} \quad E_0(f, f) \leq \| f \|_1^2. \tag{1.9}
\]

Again, by (1.3), \( E \) satisfies an inequality of the same form. Also, by (1.8) \( q_1(x, y) \leq c_1 \) for all \( x, y \), and so we can use the converse implication in [CKS, Theorem 2.9] to deduce that \( q_t(x, y) \leq c_6 t^{-n/2} \) for \( t \geq 1 \). Adjusting the constant \( c_1 \) if necessary this completes the proof of the theorem.

We now wish to use Davies’ method to obtain off-diagonal upper bounds on \( q_t \), for \( t \geq 1 \). We encounter one technical obstacle, due to the different behaviour of \( q_t \) for large and small \( t \). This means that \( E \) only satisfies a conditional Nash inequality of the form (1.9), rather than a full Nash inequality. Since verifying that the functions \( f_t \), (which arise in [CKS, Section 3]), satisfy the condition \( E(f_t, f_t) \leq \| f_t \|_2^2 \) is quite awkward, we will avoid this difficulty by using a trick.

Let \( Z = (Z_t, t \geq 0) \) be an “auxiliary” symmetric Markov process on a state space \((M, m)\), independent of \( Y \), with a transition density \( r_t(x, y) \) with respect to a \( m \) which satisfies

\[
r_t(x', y') \leq c_1 t^{-n/2}, \quad 0 < t \leq 1, \quad x', y' \in M
\]
\[
r_t(x', y') \leq c_1 t^{-n}, \quad t \geq 1, \quad x', y' \in M
\]
\[
r_t(x', x') \geq c_1 t^{-n/2}(t \vee 1)^{-n/2}, \quad t > 0.
\]

For example, if \( M \) is a sufficiently regular \( n \) dimensional manifold with volume growth given by \( V(x, r) \asymp r^{2n}, r > 1 \) and \( V(x, r) \asymp r^n, r < 1 \), we have (see for example [Gr]) that
$r_t$ satisfies (1.10). Let $X_t = (Y_t, Z_t) \in \mathbb{R}^n \times M$. Then $X$ has a transition density $p_t$ given by

$$p_t((x, x'), (y, y')) = q_t(x, y)r_t(x', y'), \quad x, y \in \mathbb{R}^n, x', y' \in M,$$

which plainly satisfies

$$\|p_t\|_\infty \leq c_1 t^{-3n/2}, \quad t > 0. \quad (1.11)$$

Write $\mathcal{E}_X$ for the Dirichlet form of $X$: $\mathcal{E}_X$ therefore satisfies the Nash inequality ($p = 3n$)

$$\|f\|_2^{2+4/p}\|f\|_1^{-4/p} \leq c_2 \mathcal{E}_X(f, f). \quad (1.12)$$

(Here $\| \cdot \|$ is of course the norm in the product space $(\mathbb{R}^n \times M, dx \times dm)$). Fix $0 \in M$. We can now use [CKS, Theorem 3.25] to deduce off-diagonal upper bounds for $p_t$. These yield immediately off-diagonal upper bounds for $q_t$, since if

$$p_t((x, 0), (y, 0)) \leq c_3 t^{-p/2}e^{-K(t, x, y)}, \quad t \geq 1,$$

then by (1.10) $q_t(x, y) \leq c_4 t^{-n/2} \exp(-K(t, x, y))$. In fact, the auxiliary process $Z$ plays no role in the calculations, and to simplify notation in what follows we will therefore omit it.

**Definition 1.3.** Set for $f \in C(\mathbb{R}^n)$

$$\Gamma(f, f)(x) = \int (f(x) - f(y))^2 n(x, y) \, dy,$$

where we allow $\Gamma(f, f) = +\infty$. Define for $\psi \in C(\mathbb{R}^n)$

$$\Lambda(\psi)^2 = \|e^{-2\psi}\Gamma(e^{\psi}, e^{\psi})\|_\infty \vee \|e^{2\psi}\Gamma(e^{-\psi}, e^{-\psi})\|_\infty,$$

$$\mathcal{F}_\infty = \{ \psi \in C(\mathbb{R}^n) : \Lambda(\psi) < \infty \},$$

$$D(t, x, y) = \sup \{ |\psi(y) - \psi(x)| - t\Lambda(\psi)^2 : \psi \in \mathcal{F}_\infty \}.$$

From [CKS, Theorem 3.25] and the remarks above, we obtain

**Lemma 1.4.** For $t \geq 1, x, y \in \mathbb{R}^n$,

$$q_t(x, y) \leq c_1 t^{-n/2} e^{-D(2t, x, y)}.$$

It remains to estimate $D(t, x, y)$.

**Lemma 1.5.** (a) For $R \geq 1$,

$$\int_{1 \leq |x-y| \leq R} |x - y|^2 n(x, y) \, dy \leq c_1.$$
(b) For $R \geq 1$, $\theta < \alpha_0/2$,

$$\int_{|x-y|>R} e^{\theta|x-y|} n(x, y) dy \leq 2c_2 e^{-(\alpha_0 - \theta)R}.$$  

Proof. Write $F(r) = \int_{|x-y|>r} n(x, y) dy$. Then $F(r) \leq c_3 e^{-\alpha r}$, by (1.1). So

$$\int_{1 \leq |x-y| \leq R} |x-y|^2 n(x, y) dy = -\int_1^R r^2 F(dr)$$

$$= F(1) - R^2 F(R) + \int_1^R 2r F(r) dr \leq c_4.$$  

Similarly,

$$\int_{|x-y|>R} e^{\theta|x-y|} n(x, y) dy = -\int_R^\infty e^{\theta r} F(dr)$$

$$\leq e^{\theta R} F(R) + \int_R^\infty c_3 \theta e^{-(\alpha_0 - \theta)r} dr$$

$$\leq c_5 e^{-(\alpha_0 - \theta)R}.$$  

\[\square\]

**Proposition 1.6.** There exist a constant $c_0$ such that if $a$ is a unit vector in $\mathbb{R}^n$, $\alpha \in (0, 1 \wedge (\alpha_0/4))$, and $\psi_\alpha(x) = \alpha a \cdot x$, then $\Lambda(\psi_\alpha)^2 \leq c_0 \alpha^2$.

Proof. We need to bound

$$e^{-2\psi_\alpha(x)} \Gamma(e^{\psi_\alpha}, e^{\psi_\alpha})(x) = \int (1 - e^{\psi_\alpha(y) - \psi_\alpha(x)})^2 n(x, y) dy. \quad (1.13)$$

Split in the integral in (1.13) into three pieces, and write

$$J_1(x) = \int_{|x-y|<1}, \quad J_2(x) = \int_{1 \leq |x-y| \leq 1/\alpha}, \quad J_3(x) = \int_{|x-y|>1/\alpha}.$$  

Then since $e^x - 1 \leq 2x$ for $0 < x < 1$,

$$J_1(x) \leq c_1 \int_{|x-y|<1} \alpha^2 |x-y|^2 n(x, y) dy$$

$$\leq c_2 \alpha^2 \int_{|x-y|<1} |x-y|^{-n+1} dy \leq c_3 \alpha^2.$$  

Similarly

$$J_2(x) \leq c_4 \int_{1 \leq |x-y| \leq 1/\alpha} \alpha^2 |x-y|^2 n(x, y) dy \leq c_5 \alpha^2,$$
by Lemma 1.5(a). Also, by Lemma 1.5(b)

\[ J_3(x) \leq \int_{|x-y| \geq 1/\alpha} e^{2\alpha|y-x|/n(x,y)} \, dy \leq c_6 e^{-\alpha_0/\alpha} \leq c_7 \alpha^2, \]

since \( \alpha < \alpha_0 / 4 \). Combining these estimates we have

\[ e^{-2\psi(x)} \Gamma(e^{\psi}, e^{\psi})(x) \leq c_0 \alpha^2, \quad x \in \mathbb{R}^n. \]

This bounds \( \|e^{-2\psi} \Gamma(e^{\psi}, e^{\psi})\|_\infty \), and replacing \( a \) by \(-a\) gives an identical bound on \( \|e^{2\psi} \Gamma(e^{-\psi}, e^{-\psi})\|_\infty \).

**Theorem 1.7.** There exists \( \alpha_1 > 0 \) such that

\[ q_t(x, y) \leq c_1 t^{-n/2} \exp(-\alpha_1 |x-y|^2 / t), \quad t \geq 1, \quad |x-y| \leq t, \quad (1.14) \]

\[ q_t(x, y) \leq c_1 t^{-n/2} \exp(-\alpha_1 |x-y|), \quad t \geq 1, \quad |x-y| \geq t. \quad (1.15) \]

**Proof.** We have, writing \( \beta = 1 \wedge (\alpha_0 / 4) \),

\[ D(2T, x, y) \geq \sup_{0 \leq \alpha \leq \beta} \left( |\psi(x) - \psi(y)| - 2T \Lambda(\psi) \right)^2 \]

\[ \geq \sup_{0 \leq \alpha \leq \beta} \left( \alpha |x-y| - 2c_0 \alpha^2 T \right). \]

If \( |x-y| \leq T \), take \( \alpha = \theta_0 |x-y| T^{-1} \) where \( \theta_0 = \beta \wedge (1/4c_0) \), to obtain

\[ D(2T, x, y) \geq \frac{1}{2} \theta_0 |x-y|^2 / T. \]

If \( |x-y| \geq T \), let \( \alpha = \theta_0 \); then

\[ D(2T, x, y) \geq \frac{1}{2} \theta_0 |x-y|. \]

The bounds (1.14), (1.15) now follow from Lemma 1.4. \( \square \)

Integrating the bounds in Theorem 1.7 we deduce

**Corollary 1.8.** There exists \( \lambda_0 < \infty \) such that for \( t \geq 1 \),

\[ \int_{|x-y| > \lambda_0 t^{1/2}} q_t(x, y) \, dy \leq \frac{1}{2}. \]

We now turn to lower bounds. The first step is to obtain a suitable Poincaré inequality. Let \( \psi \in C^\infty(\mathbb{R}, (0, \infty)) \) be such that \( \psi(x) = |x| \) for \( |x| \geq 2 \), \( \psi(x) = v(-x) \), \( |v| \leq 1 \), and \( \int e^{-v(t)} dt = 1 \). Set \( \psi(x) = \psi(x_1, \ldots, x_n) = \sum_1^n v(x_i) \), and for \( R \geq 1 \) let

\[ \varphi_R(x) = R^{-n} e^{-\psi(x/R)}. \]

Note that \( |\nabla \psi| \leq n \), and that \( \int_{\mathbb{R}^n} \varphi_R = 1 \). Write \( C \) for the set of cubes of side length 1 in \( \mathbb{R}^n \) with corners in \( \mathbb{Z}^n \) and edges parallel to the axes. If \( f \in L^1(\mathbb{R}^n) \) set

\[ f(C) = \int_C f \, dx, \quad C \in \mathcal{C}. \]

Define \( a(C, D) = 1 \) if \( C, D \) are adjacent (i.e. \( C \cap D \) is a \( n-1 \) dimensional set) and \( a(C, D) = 0 \) otherwise. From Lemma 1.19 of [SZ] we have:
Lemma 1.9. Let $g : \mathcal{C} \to \mathbb{R}$, and write

$$
\tilde{g}_R = \sum_C g(C) \varphi_R(C).
$$

Then there exists $c_1$ (independent of $R$) such that

$$
\sum_C (g(C) - \tilde{g}_R)^2 \varphi_R(C) \leq c_1 R^2 \sum_D a(C, D) (g(C) - g(D))^2 \varphi_R(C) \wedge \varphi_R(D).
$$

Let $m : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}_+$ be a measurable function such that $m(x, y) \geq 1$ whenever $|x - y|^2 \leq n + 1$. So $m(x, y) \geq 1$ if $x, y \in C$ for some $C \in \mathcal{C}$, and $m(x, y) \geq 1$ if $x \in C$, $y \in D$ and $a(C, D) = 1$.

Proposition 1.10. Let $f \in C(\mathbb{R}^n, \mathbb{R})$, and write $\overline{f}_R = \int f \varphi_R \, dx$. Then there exists $c_1$, independent of $R$, such that for $R \geq 1$,

$$
\int_{\mathbb{R}^n} (f(x) - \overline{f}_R)^2 \varphi_R(x) \, dx \leq c_1 R^2 \iint_{\mathbb{R}^n \times \mathbb{R}^n} (f(x) - f(y))^2 \varphi_R(x) \wedge \varphi_R(y) m(x, y) \, dx \, dy. \quad (1.16)
$$

Proof. From the definition of $\varphi_R$ we have that there exists $c_2 > 1$ such that

$$
c_2^{-1} \varphi_R(C) \leq \varphi_R(x) \leq c_2 \varphi_R(C) \quad \text{if} \quad x \in C. \quad (1.17)
$$

It follows that

$$
c_2^{-2} \varphi_R(D) \leq \varphi_R(C) \leq c_2^2 \varphi_R(D) \quad \text{if} \quad a(C, D) = 1.
$$

If $b \in \mathbb{R}$, then

$$
\int_{\mathbb{R}^n} (f(x) - b)^2 \varphi_R(x) \, dx = \sum_C \int_C (f(x) - b)^2 \varphi_R(x) \, dx
\leq c_3 \sum_C \varphi_R(C) \int_C (f(x) - b)^2 \, dx
= c_3 \sum_C \varphi_R(C) \int (f(x) - f(C))^2 \, dx + c_3 \sum_C \varphi_R(C) (f(C) - b)^2
= S_1 + S_2.
$$

Since

$$
\iint_{C \times C} (f(x) - f(y))^2 \, dx \, dy = 2 \int_C (f(x) - f(C))^2 \, dx,
$$

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using (1.17) we have

\[ S_1 = c_3 \sum_C \varphi_R(C) \int_C (f(x) - f(C))^2 \, dx \]

(1.18)

\[ \leq c_4 \sum_C \iint_{C \times C} (f(x) - f(y))^2 \varphi_R(C) \, dx \, dy \]

\[ \leq c_5 \sum_C \iint_{C \times C} (f(x) - f(y))^2 \varphi_R(x) \land \varphi_R(y) m(x,y) \, dx \, dy \]

\[ = c_6 \iint (f(x) - f(y))^2 \varphi_R(x) \land \varphi_R(y) m(x,y) \, dx \, dy. \]

For $S_2$, by Lemma 1.9, if $b = \tilde{f}_R = \sum_C f(C) \varphi_R(C)$,

\[ \sum_C \varphi_R(C) (f(C) - b)^2 \leq c_6 R^2 \sum_C \sum_D a(C,D) (f(C) - f(D))^2 \varphi_R(C) \land \varphi_R(D). \]

Now if $C, D \in C$,

\[ \int_C \int_D (f(x) - f(y))^2 \, dx \, dy = \int_C f^2 + \int_D f^2 - 2f(C)f(D) \]

\[ \geq (f(C) - f(D))^2. \]

So, again using (1.17),

\[ S_2 \leq c_7 R^2 \sum_C \sum_D a(C,D) \int_C \int_D (f(x) - f(y))^2 \varphi_R(x) \land \varphi_R(y) \, dx \, dy \]

(1.19)

\[ \leq c_8 R^2 \iint (f(x) - f(y))^2 \varphi_R(x) \land \varphi_R(y)m(x,y) \, dx \, dy. \]

Since $R \geq 1$, and

\[ \int (f(x) - \bar{f}_R)^2 \varphi_R(x) \, dx \leq \int (f(x) - b)^2 \varphi_R(x) \, dx, \]

combining (1.18) and (1.19) completes the proof of the Proposition.

Exactly the same argument (but with a subdivision of $\mathbb{R}^n$ into cubes of side $(n+1)^{-1/2}$) gives, using the bound (1.2) on $n(x,y)$, the following weighted Poincaré inequality.

**Theorem 1.11.** Let $f \in C(\mathbb{R}^n, \mathbb{R})$. There exists $c_1$, independent of $R$, such that for $R \geq 1$,

\[ \int (f(x) - \bar{f}_R)^2 \varphi_R(x) \, dx \leq c_1 R^2 \iint (f(x) - f(y))^2 \varphi_R(x) \land \varphi_R(y)n(x,y) \, dx \, dy. \]
We now use an argument of Fabes and Stroock [FS] (see also [SZ]) to obtain lower bounds on \( q_t(x, y) \). Let \( x_0 \in \mathbb{R}^n \), and fix \( T = R^2 \geq 1 \). Set

\[
u(t, x) = q_t(x_0, x), \quad G(t) = \int \varphi_R(x) \log u(t, x) \, dx.
\]

Then since \( u_t = \mathcal{L}_Y u \),

\[
G'(t) = \int \frac{u_t}{u} \varphi_R \, dx
= \int u^{-1} \varphi_R \mathcal{L}_Y u \, dx
= -\mathcal{E}(\varphi_R/u, u)
= -\iint \left( \frac{\varphi_R(x)}{u(t, x)} - \frac{\varphi_R(y)}{u(t, y)} \right) (u(t, x) - u(t, y)) n(x, y) \, dx \, dy.
\]

As in [SZ], using the inequality

\[
\left( \frac{d}{b} - \frac{c}{a} \right) (b - a) \leq -\frac{1}{2} (c \wedge d)(\log b - \log a)^2 + \frac{(d - c)^2}{2(c \wedge d)},
\]

(1.20)

which holds for any positive \( a, b, c, d \), we have

\[
G'(t) \geq \frac{1}{2} \iint \left( \log u(t, x) - \log u(t, y) \right)^2 \varphi_R(x) \wedge \varphi_R(y) n(x, y) \, dx \, dy
- \frac{1}{2} \iint (\varphi_R(x) - \varphi_R(y))^2 (\varphi_R(x) \wedge \varphi_R(y))^{-1} n(x, y) \, dx \, dy.
\]

(1.21)

Writing \( A(R) \) for the second term in (1.21), it follows from Theorem 1.11 that

\[
G'(t) \geq c_2 R^{-2} \int \left( \log u(t, x) - G(t) \right)^2 \varphi_R(x) \, dx - A(R).
\]

(1.22)

**Lemma 1.12.** There exists a constant \( A \in (0, \infty) \) such that \( A(R) \leq AR^{-2}, \, R \geq 1 \).

**Proof.** We have

\[
2A(R) = \int A_1(x) \, dx + \int A_2(x) \, dx + \int A_3^+(x) \, dx + \int A_3^-(x) \, dx,
\]

where

\[
A_1(x) = \int_{|x-y| \leq 1} (\varphi_R(x) - \varphi_R(y))^2 (\varphi_R(x) \wedge \varphi_R(y))^{-1} n(x, y) \, dy,
\]

and \( A_2(x), A_3^+(x), A_3^- \) are defined similarly, but with the integration over the regions

\[
\{ y : 1 \leq |x-y| \leq R \}, \{ y : |x-y| > R \} \cap \{ y : \varphi_R(y) \geq \varphi_R(x) \}, \{ y : |x-y| > R \} \cap \{ y : \varphi_R(y) < \varphi_R(x) \}
\]

respectively.
Now since \(|\nabla \psi| \leq n\), if \(|x - y| \leq R\) then \(\varphi_R(y) \geq e^{-n} \varphi_R(x)\), and
\[
|\varphi_R(x) - \varphi_R(y)| = R^{-n} e^{-\psi(x/R)}|1 - e^{\psi(x/R) - \psi(y/R)}|.
\]
Hence
\[
(\varphi_R(x) - \varphi_R(y))^2 (\varphi_R(x) \wedge \varphi_R(y))^{-1} \leq e^n \varphi_R(x) (1 - e^{\psi(x/R) - \psi(y/R)})^2
\]
\[
\leq c_1 \varphi_R(x) (e^{n|x-y|/R} - 1)^2
\]
\[
\leq c_2 \varphi_R(x) R^{-2} |x - y|^2, \quad \text{if} \quad |x - y| \leq R.
\]
So,
\[
A_1(x) \leq c_3 R^{-2} \int_{|x-y| \leq 1} \varphi_R(x) |x - y|^2 n(x, y) \, dy
\]
\[
\leq c_4 R^{-2} \varphi_R(x) \int_{1 \leq |x-y| \leq R} |x - y|^2 (\varphi_R(x) - \varphi_R(y))^2 \, dy = c_5 R^{-2} \varphi_R(x).
\]
Similarly, using (1.23) and Lemma 1.5(a)
\[
A_2(x) \leq c_6 \varphi_R(x) R^{-2} \int_{|x-y| \leq R} |x - y|^2 n(x, y) \, dy
\]
\[
\leq c_7 R^{-2} \varphi_R(x).
\]
Now writing \(B = \{ |x - y| > R \} \cap \{ \varphi_R(y) > \varphi_R(x) \}\),
\[
A_3^+(x) = \int_B \varphi_R(x)^{-1} (\varphi_R(y) - \varphi_R(x))^2 n(x, y) \, dy
\]
\[
\leq \varphi_R(x)^{-1} \int_B \varphi_R(y) n(x, y) \, dy.
\]
Since \(\varphi_R(y) \leq e^{n|x-y|/R} \varphi_R(x)\), if \(R > 2n/\alpha_0\) then
\[
A_3^+(x) \leq \varphi_R(x) \int_{|y-x| > R} e^{n|x-y|/R} n(x, y) \, dy
\]
\[
\leq c_8 \varphi_R(x) e^{-\alpha_0 R + n} = c_9 e^{-\alpha_0 R} \varphi_R(x).
\]
By symmetry \(\int A_3^+(x) \, dx = \int A_3^-(x) \, dx\), so combining the estimates (1.24)–(1.26),
\[
A(R) \leq c_{10} \left( R^{-2} + e^{-\alpha_0 R} \right) \int \varphi_R(x) \, dx \leq c_{11} R^{-2},
\]
which proves the lemma. \(\square\)
Lemma 1.13. Let $T, R, G(t), x_0$ be as above. Then there exists a constant $c_1$ such that
\[ G(T) \geq -c_1 + \log(T^{-n/2}) \quad \text{provided} \quad |x_0| \leq R. \quad (1.27) \]

Proof. Set $u_0(s, x) = R^n u(sT, x)$, and
\[ G_0(s) = \int \varphi_R(x) \log u_0(s, x) \, dx = G(sT) + \log R^n. \]

Then for $0 < s < 1$, using (1.22) and Lemma 1.12,
\[ T G_0'(sT) \geq -A + c_1 \int (\log u(Ts, x) - G(Ts))^2 \varphi_R(x) \, dx \]
\[ = -A + c_1 \int (\log u_0(s, x) - G_0(s))^2 \varphi_R(x) \, dx. \]

We can now follow very closely the argument of [FS, Lemma 2.1]. By (1.7) we have
\[ \sup_{\frac{1}{2} \leq s \leq 1} u_0(s, x) \leq K, \]
and so, since $(\log u_0 - G_0)^2 u_0^{-1} \geq (\log K - G_0)^2 K^{-1}$ when $u_0 \geq e^{2+G_0}$ we have
\[ G_0'(s) \geq -A + c_2 \int_{u_0(s, x) \geq e^{2+G_0(s)}} \varphi_R(x) u_0(s, x) \, dx. \]

Let $\theta \geq 1$. Then for $\frac{1}{2} \leq s \leq 1$,
\[ \int_{u_0(s, x) \geq e^{2+G_0(s)}} \varphi_R(x) u_0(s, x) \, dx \geq \int \varphi_R(x) u_0(s, x) \, dx - e^{2+G_0(s)} \]
\[ \geq \int_{|x| < \theta R} \varphi_R(x) u_0(s, x) \, dx - e^{2+G_0(s)} \]
\[ \geq R^n \inf_{|x| < \theta R} \varphi_R(x) \left( 1 - \int_{|x| > \theta R} u(Ts, x) \, dx \right) - e^{2+G_0(s)}. \]

Now
\[ R^n \inf_{|x| < \theta R} \varphi_R(x) = \inf_{|y| \leq \theta} e^{-\psi(y)} \geq e^{-n\theta}, \]
while by Corollary 1.8, if $\theta$ is chosen large enough,
\[ \int_{|x| > \theta R} u(Ts, x) \, dx \leq \frac{1}{2} \quad \text{for} \quad \frac{1}{2} \leq s \leq 1. \]

We can now proceed, exactly as in [FS], to deduce that $G_0'(s)$ satisfies a differential inequality which implies that $G_0'(1) \geq -c_1$. (1.27) is then immediate. \qed
Theorem 1.14. There exists a constant $a_1$ such that
\[ q_t(x, y) \geq c_1 t^{-n/2} \quad \text{for} \quad t \geq 2, \quad |x - y| \leq a_1 t^{1/2}. \] (1.28)

Proof. It is sufficient to prove this for $x = 0$. Write $T = t/2$, $R = T^{1/2}$. Since
\[ q_{2T}(0, y) = \int q_T(0, x)q_T(x, y) \, dx \]
\[ \geq \int q_T(0, x)q_T(x, y)R^n \varphi_R(x) \, dx, \]
then by Jensen’s inequality,
\[ \log T^{-n/2}q_T(0, y) \geq \int (\log q_T(0, x)) \varphi_R(x) \, dx + \int \varphi_R(x) \log q_T(x, y) \, dx. \]
So if $|y| < T^{1/2}$, from Lemma 1.13,
\[ \log T^{-n/2}q_{2T}(0, y) \geq -2c_1 + 2 \log T^{-n/2}, \]
which establishes (1.28).

We can now obtain lower bounds for $q$ from (1.28) by a chaining argument. We omit the proof, as the argument is standard and the bound (1.28) is already sufficient to establish the Liouville property for $Y$.

Theorem 1.15. There exist constants $c_i$ such that
\[ q_t(x, y) \geq c_1 t^{-n/2} \exp(-c_2|x - y|^2/t), \quad t \geq 2, \quad |x - y| \leq c_3 t. \]

Definition 1.16. Write $(Q_t, t \geq 0)$ for the semigroup of $Y$: $Q_t f(x) = Q^x f(Y_t)$. A bounded function $h$ is $Y$-harmonic if $Q_t h = h$ for all $t \geq 0$, or equivalently, if $h(Y_t)$ is a martingale$/Q^x$ for all $x$.

Theorem 1.17. Let $h$ be bounded and $Y$-harmonic. Then $h$ is constant.

Proof. Suppose $h$ is non-constant. Replacing $h$ by $ah + b$ if necessary, we can assume that $\inf h = 0$, $\sup h = 1$. So there exists $x_0 \in \mathbb{R}^n$ such that $h(x_0) \geq \frac{3}{4}$. We can assume $x_0 = 0$. Let $\lambda_0$ be as in Corollary 1.8, and write $B(t) = B(0, \lambda_0 t^{1/2})$ for $t \geq 1$. Then since $h(0) = Q_t h(0)$, and
\[ \int_{B(t)^c} q_t(0, y)h(y) \, dy \leq \frac{1}{2}, \quad t \geq 1, \]
we must have
\[ \int_{B(t)} q_t(0, y)h(y) \, dy \geq \frac{1}{4}, \quad t \geq 1. \]
Lemma 2.1. For \( \delta > 0, x' \in \mathbb{R}^n \) set
\[
C'(x', \delta) = \{(y', y_d) : |y' - x'| < \delta\},
\]
\[
C_0(x', \delta) = C'(x', \delta) \cap \{(y', y_d) : -K_1 - \delta \leq y_d - \gamma(x') \leq K_1 + \delta\}.
\]
Then there exists \( \delta_0 = \delta_0(\varepsilon_0, K_0, K_1, K_2) > 0 \) such that for \( x' \in \mathbb{R}^n \)
\[
C_0(x', \delta_0) \subset I(-K_2 + \delta_0, K_2 - \delta_0),
\]
and
\[
H(0) \cap C_0(x', \delta_0) = H(0) \cap C(x', \delta_0).
\]
Let
\[
\mathcal{L} = \frac{1}{2} \sigma^{-2} \nabla (\sigma^2 \nabla) = \frac{1}{2} \Delta + (\sigma^{-1} \nabla \sigma) \nabla.
\]

2. Probabilistic estimates and Girsanov transformation.

For \( x \in \mathbb{R}^d = \mathbb{R}^n \times \mathbb{R} \) we write \( x = (x', x_d) \) where \( x' = (x_1, \ldots, x_n) \in \mathbb{R}^{d-1} \). Let \( \gamma : \mathbb{R}^n \to \mathbb{R} \) be \( C^2 \), with
\[
|\nabla \gamma(x')| \leq K_0, \quad x' \in \mathbb{R}^n.
\]
For \(-\infty \leq a \leq b \leq \infty \) set
\[
I(a, b) = \{(x', x_d) \in \mathbb{R}^d : \gamma(x') + a \leq x_d \leq \gamma(x') + b\},
\]
\[
H(\lambda) = I(\lambda, \lambda), \quad \lambda \in \mathbb{R}.
\]

Let \( \sigma : \mathbb{R}^d \to (0, \infty) \) be a smooth function. We assume that there exist constants
\( 0 < \varepsilon_0 < 1, 1 \leq K_1 < K_2 < \infty, K_3 < \infty \) such that \( \sigma \) satisfies
\[
\text{(S1)} \quad \sigma^{-1} \Delta \sigma \geq 2\varepsilon_0 \text{ on } I(-K_1, K_1)^c,
\]
\[
\text{(S2)} \quad \sigma \geq \varepsilon_0/2 \text{ on } I(-K_2, K_2),
\]
\[
\text{(S3)} \quad ||\sigma||_\infty, ||\nabla \sigma||_\infty \text{ and } ||\Delta \sigma||_\infty \text{ are all bounded by } K_3.
\]

We will require the following easy geometric property of the sets \( I(a, b) \).

Let \( x' \in \mathbb{R}^n \). Choose \( t \) large enough so \( x \in B(t) \), and so that \( t^{1/2} \geq \lambda_0 \). Then for \( y \in B(t) \),
\[
q_t(0, y) \leq c_1 t^{-n/2} \exp(-c_2 \lambda_0).
\]
But if \( s = 2\lambda_0 t/a_1 \), by Theorem 1.14
\[
q_s(x, y) \geq c_3 s^{-n/2} = c_4 t^{-n/2} \geq c_5 q_t(0, y), \quad y \in B(R),
\]
where \( c_5 > 0 \). Thus
\[
h(x) = \int q_s(x, y)h(y) \, dy \geq c_5 \int_{B(t)} q_t(x, y)h(y) \, dy \geq \frac{1}{4} c_5.
\]
So \( \inf h \geq c_5/4 \), a contradiction. \( \square \)
We use Girsanov’s theorem to construct a diffusion with generator $\mathcal{L}$. Let $X = (X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$ be a standard Brownian motion on $\mathbb{R}^d$ defined on a probability space $(\Omega, \mathcal{F})$ with filtration $(\mathcal{F}_t)$. Set

$$ U_t = \int_0^t \nabla (\log \sigma)(X_s) dX_s - \frac{1}{2} \int_0^t |\nabla \log \sigma(X_s)|^2 ds. $$

By Itô’s formula, since

$$ \log \sigma(X_t) = \log \sigma(X_0) + \int_0^t \nabla (\log \sigma)(X_s) dX_s + \frac{1}{2} \int_0^t \Delta (\log \sigma)(X_s) ds, $$

and $\Delta \log \sigma + |\nabla \log \sigma|^2 = \sigma^{-1} \Delta \sigma$, we have

$$ U_t = \log \sigma(X_t) - \log \sigma(X_0) - \frac{1}{2} \int_0^t (\sigma^{-1} \Delta \sigma)(X_s) ds. \quad (2.2) $$

Write $V = \frac{1}{2} \sigma^{-1} \Delta \sigma$, and set

$$ Z_t = \exp(U_t) = \sigma(X_t) \sigma(X_0)^{-1} \exp \left( -\frac{1}{2} \int_0^t (\sigma^{-1} \Delta \sigma)(X_s) ds \right) $$

$$ = \sigma(X_t) \sigma(X_0)^{-1} \exp \left( - \int_0^t V(X_s) ds \right). $$

Note that $u > \varepsilon_0$ on $I(-K_1, K_1)^c$, and $|u| \leq K_3/\varepsilon_0$ on $I(-K_2, K_2)$, so that $-u \leq K_3/\varepsilon_0$ everywhere.

**Lemma 2.2.** (a) If $X_s \in I(-K_1, K_1)^c$ for $0 \leq s \leq t$, then

$$ \sup_{s \leq t} Z_s \leq \sigma(X_0)^{-1} \sigma(X_t) e^{-\varepsilon_0 t} \leq \sigma(X_0)^{-1} K_3 e^{-\varepsilon_0 t}. $$

(b) If $X_s \in I(-K_2, K_2)$ for $0 \leq s \leq t$, then

$$ Z_t^{-1} \leq \frac{1}{2} K_3^{-1} \varepsilon_0 e^{-K_3 t/\varepsilon_0} \leq Z_t. $$

(c) $Z$ satisfies

$$ \sup_{s \leq t} Z_s \leq \sigma(X_0)^{-1} K_3 e^{K_3 t/\varepsilon_0}. $$

(d) For each $x \in \mathbb{R}^d$, $Z$ is a martingale with respect to $\mathbb{P}^x$.

**Proof.** (a), (b) and (c) are immediate from the definition of $Z$, and the properties of $\sigma$ and $V$.

(d) $Z$ is a local martingale, since $Z$ is of the form $Z_t = \exp(M_t - \frac{1}{2} \langle M \rangle_t)$, where $M$ is a continuous local martingale. But then $Z$ is a true martingale, since (c) implies that $Z$ is $\mathbb{P}^x$-a.s bounded on every interval $[0, t]$. \qed
We can now use Girsanov's transformation (see for example [RW, Theorem 38.9]) to define a probability measure $\tilde{P}^x$ on $(\Omega, \mathcal{F})$ such that $d\tilde{P}^x/dP^x \big|_{\mathcal{F}_t} = Z_t$. Then under $\tilde{P}^x$,

$$X_t - \int_0^t \nabla \log \sigma(X_s) \, ds = W_t, \quad (2.3)$$

where $W$ is a Brownian motion with respect to $\tilde{P}^x$ with $W_0 = x$. So, under $\tilde{P}^x$, $X$ is a diffusion with generator $\mathcal{L}$ given by (2.1). Define

$$\tau_\lambda = \inf\{s \geq 0 : X_s \in H(\lambda)\}.$$

**Lemma 2.3.** (a) If $\lambda \geq K_1$, $y \in I(\lambda, \infty)$ then $\tilde{P}^y(\tau_\lambda < \infty) = 1$.
(b) For any $y \in \mathbb{R}^d$, $\tilde{P}^y(\tau_0 < \infty) = 1$.
(c) For $x \in H(0)$, $\tilde{P}^x(\tau_{K_2} < \infty) = 1$.

**Proof.** (a) Let $t \geq 0$. By the definition of $\tilde{P}^y$, and Lemma 2.2(a),

$$\tilde{P}^y(\tau_\lambda > t) = \mathbb{E}^y 1_{(\tau_\lambda > t)} Z_t \leq \sigma(y)^{-1} K_3 e^{-\varepsilon_0 t} \tilde{P}^y(\tau_\lambda > t) \leq \sigma(y)^{-1} K_3 e^{-\varepsilon_0 t}.$$

Letting $t \to \infty$ (a) is immediate.
(b) Let $x = (x', x_d) \in I(-K_1, K_1)$, and set

$$F = \{X_s \in C_0(x', \delta_0), 0 \leq s \leq 1\} \cap \{\tau_0 < 1\}.$$

By the support theorem for Brownian motion (see [Bs1, p.25]) there exists $p_0 > 0$ (independent of $x$) such that $P^x(F) \geq p_0$. By Lemma 2.1 the cylinder $C_0(x', \delta_0) \subset I(-K_1, K_1)$, so using Lemma 2.2(b)

$$\tilde{P}^x(F) = \mathbb{E}^x 1_F Z_1 \geq \frac{1}{2} K_3^{-1} \varepsilon_0 e^{K_3/\varepsilon_0} p_0.$$

Using this and (a), if $x \in I(0, \infty)$, then a standard renewal argument implies that $\tilde{P}^x(\tau_0 < \infty) = 1$. Exactly the same argument works for $x \in I(-\infty, 0)$.
(c) This is proved, using the support theorem, by an argument similar to the above. \[\square\]

The main result of this section is an exponential moment bound on $|X_{\tau_0} - x|$ for $x \in I(-K_1, K_1)$, under $\tilde{P}^x$. As in Lemma 2.3, it is enough to treat the case $x \in I(0, K_1)$. Define stopping times

$$T_0 = 0,$$
$$S_n = \min\{t \geq T_{n-1} : X_t \in H(0) \cup H(K_2)\}, \quad n \geq 1,$$
$$T_n = \min\{t \geq S_n : X_t \in H(K_1)\}, \quad n \geq 1,$$

and let

$$N = \min\{n \geq 1 : X_{S_n} \in H(0)\}.$$
These random variables are all $\tilde{\mathbb{P}}^x$-a.s. finite by Lemma 2.3. Set

$$\xi_n = |X_{S_n} - X_{T_{n-1}}|, \quad \eta_n = |X_{T_n} - X_{S_n}|, \quad n \geq 1.$$  

Clearly

$$|X_{T_0} - x| \leq \sum_{n=1}^N \xi_n + \sum_{n=1}^{N-1} \eta_n. \quad (2.5)$$

Note that if $x \in H(K_2)$, then $\tilde{\mathbb{P}}^x(T_0 = S_1 = 0) = 1$.

**Lemma 2.4.** There exist $c_0, c_1$ such that

$$\tilde{\mathbb{P}}^x(\eta_n > \lambda | \mathcal{F}_{S_n})1_{(n < N)} \leq c_0 e^{-c_1 \lambda}. \quad (2.6)$$

**Proof.** Using the strong Markov property of $X$, it is enough to prove

$$\tilde{\mathbb{P}}^y(\eta_1 > \lambda) \leq c_0 e^{-c_1 \lambda}, \quad y \in H(K_2).$$

So let $y \in H(K_2)$. Then $\eta_1 = |X_{T_1} - y|$ and

$$\tilde{\mathbb{P}}^y(\eta_1 > \lambda) \leq \tilde{\mathbb{P}}^y(T_1 > \lambda) + \tilde{\mathbb{P}}^y(T_1 \leq \lambda, \eta_1 > \lambda). \quad (2.7)$$

Using (2.4), we have

$$\tilde{\mathbb{P}}^y(T_1 > \lambda) \leq c_2 e^{-c_0 \lambda}.$$  

For the second term in (2.7), note that by Lemma 2.2(a) $Z_{\lambda \wedge T_1} \leq c_3 e^{-\lambda \varepsilon_0}$, so that

$$\tilde{\mathbb{P}}^y(\eta > \lambda, T_1 \leq \lambda) \leq c_4 e^{-c_0 \lambda} \tilde{\mathbb{P}}^y(\sup_{0 \leq s \leq \lambda} |X_s - y| > \lambda)$$

$$\leq c_5 \exp \left(-c_0 \lambda - c_6 \lambda\right),$$

where we used a standard bound on Brownian motion in the last line. Combining these estimates for the two terms in (2.7) proves the lemma. $\square$

**Lemma 2.5.** There exist $\delta_1 > 0, c_1, c_2 < \infty$ such that

$$\tilde{\mathbb{P}}^x(\xi_n > \lambda | \mathcal{F}_{T_{n-1}})1_{(N > n-1)} \leq c_1 e^{-c_2 \lambda}, \quad \lambda > 0, \quad (2.8)$$

$$\tilde{\mathbb{P}}^x(X_{S_n} \in H(0) | \mathcal{F}_{T_{n-1}})1_{(N > n-1)} \geq \delta_1. \quad (2.9)$$

**Proof.** As in the previous lemma, it is sufficient to take $x \in H(K_1)$ and prove unconditional versions of (2.8) and (2.9) with $n = 1$.

The estimate (2.9) follows from the support theorem for Brownian motion by the same argument as in Lemma 2.3(b).

Let $\sigma_1 \in C^2(\mathbb{R}^d)$ be defined by taking $\sigma_1 = \sigma$ in $I(-K_2, K_2)$, and be such that $\frac{1}{3} \varepsilon_0 \leq \sigma_1(y) \leq 2K_3$ for $y \in \mathbb{R}^d$. Let $X^* = (X^*_t, t \geq 0, \mathbb{Q}^x, x \in \mathbb{R}^d)$ be the divergence form diffusion with generator $L^* = \frac{1}{2} \nabla \sigma_1^2 \nabla$. Then $X_s, s \in [0, T_1]$, is, under $\tilde{\mathbb{P}}^x$, a time change of $X^*$, and so $\tilde{\mathbb{P}}^x(\xi_1 > \lambda) = \mathbb{Q}^x(|X^*_R - x| > \lambda)$, where $R = \inf\{t \geq 0 : X^*_t \in H(0) \cup H(K_2)\}$.

The bound (2.8) now follows from standard properties of uniformly elliptic divergence form diffusions; see [BBu], Lemma 2.2. and Section 4 and [Bs2], pp. 187–188. $\square$
Theorem 2.6. There exist constants $c_0, c_1$, such that
\[ \tilde{\mathbb{P}}^x(|X_{\tau_0} - x| > \lambda) \leq c_0 e^{-c_1 \lambda}, \quad \lambda > 0, \quad x \in I(-K_2, K_2). \]

Proof. Set $V_1 = \xi_1$, and
\[ V_n = |X_{S_n - S_{n-1}}| 1_{(N>n-1)} \leq (\xi_n + \eta_{n-1}) 1_{(N>n-1)}, \quad n \geq 2. \]
Combining (2.6) and (2.8) we deduce that there exists $\alpha > 0$ such that
\[ \tilde{\mathbb{P}}^x(V_n > \lambda \mid \mathcal{F}_{S_{n-1}}) \leq c_1 e^{-\alpha \lambda}, \quad \lambda > 0. \]
Integrating this bound, for $\theta < \alpha$,
\[ \tilde{\mathbb{E}}^x(e^{\theta V_n} \mid \mathcal{F}_{S_{n-1}}) \leq 1 + \frac{c_1 \theta}{\alpha - \theta}. \]
Write $\psi(\theta) = \log(1 + \frac{c_1 \theta}{\alpha - \theta})$. Then if $M_n = \exp\left(\theta \sum_{i=1}^{N \wedge k} V_i - n \psi(\theta)\right)$,
\[ \tilde{\mathbb{E}}^x(M_n \mid \mathcal{F}_{S_{n-1}}) = M_{n-1} e^{-\psi(\theta)} \tilde{\mathbb{E}}^x(e^{\theta V_n} \mid \mathcal{F}_{S_{n-1}}) \leq M_{n-1}, \]
so that $M_n$ is a supermartingale. Since $N$ is a stopping time with respect to $(\mathcal{F}_n)$, if $k \geq 1$ then $\tilde{\mathbb{E}}^x(M_{N \wedge k}) \leq 1$. So, by Cauchy-Schwarz
\[ \tilde{\mathbb{E}}^x \exp\left(\frac{1}{2} \theta \sum_{i=1}^{N \wedge k} V_i\right) = \tilde{\mathbb{E}}^x \left(\exp\left(\frac{1}{2} \theta \sum_{i=1}^{N \wedge k} V_i - \frac{1}{2} (N \wedge k) \psi(\theta)\right) \exp\left(\frac{1}{2} (N \wedge k) \psi(\theta)\right)\right) \]
\[ \leq \left(\tilde{\mathbb{E}}^x \exp\left(\theta \sum_{i=1}^{N \wedge k} V_i - (N \wedge k) \psi(\theta)\right)\right)^{1/2} \left(\tilde{\mathbb{E}}^x e^{\psi(\theta)(N \wedge k)}\right)^{1/2} \]
\[ \leq \left(\tilde{\mathbb{E}}^x e^{\psi(\theta)N}\right)^{1/2}. \]
The bound (2.9) implies that $P(N = n \mid N > n - 1) \geq \delta_1$, so $P(N \geq n) \leq (1 - \delta_1)^{n-1}$, and
\[ \tilde{\mathbb{E}}^x(e^{\psi(\theta)N}) < \infty \quad \text{provided} \quad e^{\psi(\theta)(1 - \delta_1)} < 1. \]
So, taking $\theta_1$ small enough so that this last condition holds, and letting $k \to \infty$, it follows that
\[ \tilde{\mathbb{E}}^x \exp\left(\frac{1}{2} \theta \sum_{i=1}^{N} V_i\right) < \infty \quad \text{for} \quad \theta < \theta_1. \]
Since
\[ |X_{\tau_0} - X_0| = |X_{S_N} - X_0| \leq \sum_{i=1}^{N} V_i, \]
this implies that $|X_{\tau_0} - X_0|$ has exponential moments with respect to $\tilde{\mathbb{P}}^x$, proving the theorem.

The final result in this section will be used to weaken the hypotheses of boundedness in our Liouville Theorem.
Proposition 2.7. Let $h \in C^2(\mathbb{R}^d)$ be a function such that $\mathcal{L}h = 0$ and $\sigma h$ is bounded. Then $h$ is bounded.

**Proof.** We can assume that $|\sigma h| \leq 1$. Set $M_t = h(X_t)$. By Itô’s formula $M$ is a continuous local martingale with respect to $\mathbb{P}^x$, and $M_t Z_t$ is a continuous local martingale with respect to $\mathbb{P}^x$. However,

$$|M_t Z_t| = \sigma(X_0)^{-1} |\sigma(X_t)h(X_t)| \exp \left( - \int_0^t V(X_s)ds \right) \leq \sigma(X_0)^{-1} e^{tK_3/\varepsilon_0},$$

so that $|M_t Z_t|$ is bounded on each interval $[0,t]$. Therefore $MZ$ is a martingale with respect to $\mathbb{P}^x$, and hence $M$ is a martingale with respect to $\mathbb{P}^x$.

Set $T = \inf\{t \geq 0 : X_t \in I(-K_2, K_2)\}$; by Lemma 2.3 $\mathbb{P}^x(T < \infty) = 1$ for all $x$. Note that, as $h$ is bounded by $2/\varepsilon_0$ on $I(-K_2, K_2)$, $|h(X_T)| \leq 2/\varepsilon_0$. Since $M$ is a martingale with respect to $\mathbb{P}^x$, $\mathbb{E}^x h(X_T \wedge T) = h(x)$. So

$$|\mathbb{E}^x h(X_T) - h(x)| = |\mathbb{E}^x 1_{(T > t)}(h(X_T) - h(X_t))| \leq 2\varepsilon_0^{-1} \mathbb{E}^x(T > t) + \mathbb{E}^x 1_{(T > t)}|h(X_t)|. \tag{2.10}$$

By Lemma 2.2(a) the second term in (2.10) is bounded by

$$\sigma(x)^{-1} \mathbb{E}^x 1_{(T > t)} \sigma(X_t)|h(X_t)|e^{-\varepsilon_0 t} \leq \sigma(x)^{-1} e^{-\varepsilon_0 t}.$$

Since $\mathbb{P}^x(T = \infty) = 0$, letting $t \to \infty$ in (2.10) it follows that $\mathbb{E}^x h(X_T) = h(x)$. Hence $|h(x)| \leq \mathbb{E}^x|h(X_T)| \leq 2/\varepsilon_0$, proving that $h$ is bounded. \hfill \square

3. Transformation to a Jump Process

We continue with the notation and hypotheses of the previous section. We write $X = (X_t, t \geq 0, \mathbb{P}^x, x \in \mathbb{R}^d)$ for the diffusion with generator $\mathcal{L}$. When we refer to properties of $X$, it is with respect to the probabilities $\mathbb{P}^x$. Let $\mu(dx) = \sigma^2(x)dx$, and define

$$\mathcal{E}(f, f) = \int |\nabla f(x)|^2 \sigma^2(x)dx, \quad f \in C_0^2(\mathbb{R}^d).$$

Then (see [FOT, Thm. 3.1.3]) $\mathcal{E}$ can be extended to a regular Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ on $L^2(\mathbb{R}^d, \mu)$ with (special standard) core $C_0^2(\mathbb{R}^d)$, and $X$ is the Hunt process (in fact a Feller diffusion) associated with $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$.

Define $\pi : H(0) \to \mathbb{R}^{d-1}$ by $\pi((x', x_d) = x'$. Let $\nu$ be the measure on $\mathbb{R}^d$ with support $H(0)$ given by

$$\nu(A \cap H(0)) = |\pi(A \cap H(0))|,$$

where $|\cdot|$ denotes $d-1$–dimensional Lebesgue measure. Let $A_t$, $t \geq 0$ be the continuous additive functional with Revuz measure $\nu$. Note that $A_t$ increases only when $X_t \in H(0)$. Set $\zeta_t = \inf\{s \geq 0 : A_s > t\}$, and let

$$\tilde{X}_t = X_{\zeta_t}.$$
It is clear from Lemma 2.3 that $\tilde{P}_t^x(\zeta_t < \infty) = 1$ for all $t$, so that $\tilde{X}$ is a conservative Markov process.

We now use the relation between traces of Dirichlet forms and time changes of Hunt processes – see [FOT], Theorem 6.2.1, to describe the Dirichlet form of $\tilde{X}$. Let

$$
\Gamma(x, dy) = \tilde{P}_x^x(X_{\tau_0} \in dy), \quad x \in \mathbb{R}^d - H(0), \; y \in H(0),
$$

be the harmonic measure on $H(0)$ for $X$. Since $X$ is a diffusion with $C^2$ coefficients, $\Gamma$ is absolutely continuous with respect to $\nu$. Write $\Gamma(x, y)$ for the density of $\Gamma$:

$$
\Gamma(x, dy) = \Gamma(x, y) \nu(dy), \quad y \in H(0), \; x \in \mathbb{R}^d - H(0).
$$

Further $\Gamma(x, y)$ is continuous on $(\mathbb{R}^d - H(0)) \times H(0)$. For $g \in C^2_0(\mathbb{R}^d)$, set

$$
\Gamma g(x) = \begin{cases} 
g(x), & \text{if } x \in H(0), \\
\int \Gamma(x, dy) g(y), & \text{if } x \in \mathbb{R}^d - H(0).
\end{cases}
$$

Then $\Gamma g$ is $\mathcal{L}$-harmonic on $H(0)^c$, and continuous on $\mathbb{R}^d$. By [FOT, Theorem 6.2.1] $\tilde{X}$ is a $\nu$-symmetric Hunt process with Dirichlet form $(\tilde{\mathcal{E}}, \mathcal{D}(\tilde{\mathcal{E}}))$ with core $C^2_0(H(0))$, where $\tilde{\mathcal{E}}$ is given by

$$
\tilde{\mathcal{E}}(g, g) = \mathcal{E}(\Gamma g, \Gamma g), \quad g \in C^2_0(H(0)). \tag{3.1}
$$

**Remark 3.1.** Let $W$ be $BM(\mathbb{R}^d)$; $W$ has Dirichlet form

$$
\mathcal{E}_B(f, f) = \int |\nabla f(x)|^2 dx.
$$

If $H(0) = \{x : x_d = 0\}$, then a similar construction gives a Dirichlet form $\tilde{\mathcal{E}}_B$. See [FOT, Example 6.2.2] for a detailed discussion of this case: $\tilde{\mathcal{E}}_B$ is the Dirichlet form of the $d - 1$-dimensional Cauchy process, and is given by

$$
\tilde{\mathcal{E}}_B(g, g) = c_d \int_{H(0) \times H(0)} (g(x) - g(y))^2 |x - y|^{-d} \, dx \, dy. \tag{3.2}
$$

Since the law of $X$ is locally absolutely continuous with respect to that of Brownian motion, one expects an analogous expression for $\tilde{\mathcal{E}}$.

We now wish to identify more precisely the Dirichlet form $\tilde{\mathcal{E}}$. We first show there exists a nice harmonic function to compare to.

**Lemma 3.2.** There exist $c_1$ and $c_2$ and an $\mathcal{L}$-harmonic function $h$ such that $h = 0$ on $H(0)$, $h = 1$ on $H(1)$, $|\nabla h| \leq c_1$ in $I(0, 1)$, and $\partial h / \partial n \geq c_2$ on $H(0)$.

**Proof.** Let $h(x) = \mathbb{P}^x(\tau_1 < \tau_0)$. Since $h$ solves a Dirichlet problem in a $C^2$ domain with $C^2$ boundary values and $C^2$ strictly elliptic coefficients for the operator, the upper bound on $|\nabla h|$ follows. We need only prove the lower bound.


We flatten the boundary. That is, we look at $\tilde{X}_t = \Phi(X_t)$, where $\Phi(x', x_d) = (x', x_d - \gamma(x'))$. A routine calculation using Ito’s formula shows that on $\Phi(I(0, 1))$ the process $\tilde{X}_t$ is associated with an operator in divergence form that is strictly elliptic, that the coefficients are $C^2$, and that the normal derivative gets mapped into the conormal derivative; also the angle made by the conormal vector with the hyperplane is bounded away from 0. If $\tilde{h}$ is the image of $h$ under this map, we need to show that $\partial \tilde{h}/\partial n > c_3$.

Since the coefficients of the operator are $C^2$, the process $\tilde{X}_t^d$ is a semimartingale $M_t + A_t$, where $c_4 \leq d(M)_{1}/dt \leq c_5$ and $|dA_t| \leq c_0 dt$. By performing a nondegenerate time change, we may assume $\tilde{M}_t$ is a Brownian motion. By a comparison theorem, (see, e.g., [IW], p. 352)

$$\tilde{h}(x) \geq \mathbb{P}^x(W_t - c_7 t \text{ hits } 1 \text{ before } 0),$$

where $W_t$ is a standard one-dimensional Brownian motion. This, it is well known, is larger than $c_8 x_d$. \hfill \Box

Next we need some routine harmonic measure estimates. For $x \in H(0)$ we let $B_{H(0)}(x, r)$ be $\{y \in H(0) : |y - x| < r\}$, $G$ be the Green function for the process killed on hitting $H(0)$, and $y_r(x)$ the point whose coordinates are the same as those for $x$, except that the $x_d$ coordinate is $r$ larger; thus $y_r(x)$ lies $r$ units above $x$. Since $K_1 \geq 1$ $\mathcal{L}$ is strictly elliptic on $I(0, 1)$.

**Proposition 3.3.** Suppose $x_0 \in I(0, 1)$ with dist $(x_0, H(0)) \geq 1/4$. Let $x \in H(0)$ with $|x - x_0| \leq 2$. Then there exist $c_1, c_2, c_3, c_4$, and $A_0$ not depending on $x_0$ or $x$ such that

(a) $c_1 \leq \Gamma(x_0, y) \leq c_2$ if $|y - x| \leq 1$.

(b) If $\lambda \geq A_0$, then $\mathbb{P}^{x_0}(|X_{\tau_0} - x| > \lambda) \leq c_3 \exp(-c_4 \lambda)$.

**Proof.** Suppose $x_0 \in I(1/2, 1)$. As in [Bs1], Theorem III.5.4,

$$\Gamma(x_0, B_{H(0)}(x, r)) \approx G(x_0, y_r(x)) r^{d-2}, \quad r < \frac{1}{8}.$$ 

Here ‘$\approx$’ means that the ratio is bounded above and below by positive constants not depending on $x$ or $r$ as long as $r < 1$. The proof in [Bs1] is given for Brownian motion, but the identical proof works for strictly elliptic divergence form operators. We now apply the the boundary Harnack principle for divergence from operators (see [BBu]) to compare the harmonic functions $h(y)$ and $G(x_0, y)$. We conclude $G(x_0, y_r(x)) \approx h(y_r(x))$ and hence $G(x_0, y_r(x)) \approx r$. So $\Gamma(x_0, (B(x, r)) \approx r^{d-1}$, as long as $r < \frac{1}{8}$ and $|x - x_0| < 2$. Since the surface measure of $B(x, r)$ is comparable to $r^{d-1}$, (a) follows.

(b) follows immediately from Theorem 2.6. \hfill \Box

Now we estimate $\partial \Gamma/\partial n$. Let $S$ be the surface measure on $H(0)$; note that $S$ and $\nu$ are absolutely continuous. Since $\Gamma$ is a solution to a Dirichlet problem, $\partial \Gamma/\partial n$ exists. Let $m(x, y) = \frac{\partial \Gamma(\cdot, y)}{\partial n}(x)$.

**Proposition 3.4.** There exist $c_1, c_2, c_3, c_4$, and $A_0$ such that for $x, y \in H(0)$,
(a) If \( |x - y| \leq 1 \), then
\[ c_1|x - y|^d \leq m(x, y) \leq c_2|x - y|^d. \]

(b) If \( A > A_0 \), then
\[ \int_{B_{H(0)}(x, A)^c} m(x, y) S(dy) \leq c_3 \exp(-c_4 A). \]

Proof. Let us first flatten the boundary as above. Pick \( z \in I(0, 1) \) with \( z' = x' \).

First suppose \( |x - y| = 1 \). By the boundary Harnack principle, \( \Gamma(z, y) \) is comparable to
\[ \frac{h(z)}{h((z', 1/2))} \Gamma((z', 1/2), y). \]
This is comparable to \( z_d \) by Lemma 3.2 and Proposition 3.3. (a) follows when \( |x - y| = 1 \) by letting \( z_d \to 0 \). (Recall that the angle between the conormal vector and the hyperplane is bounded away from 0.)

We get the case where \( |x - y| < 1 \) by scaling. Let \( r = |x - y| \) and scale by a factor \( 1/r \). This enlarges things, so the region on which \( \sigma \) is nice is larger and the coefficients are smoother. This increases the area of a surface ball by a factor \( r^{d-1} \), and since the distance from \( z \) to the boundary becomes \( r \) times as large, the derivative increases by a factor \( r \). So altogether we get a factor \( r^d \), as we should.

To get (b), we use the boundary Harnack principle as above. So \( \Gamma(z, (B_{H(0)}(x, A)^c) \) is comparable to
\[ \frac{h(z)}{h((z', 1/2))} \Gamma((z', 1/2), (B_{H(0)}(x, A)^c). \]
By Proposition 3.3, this is less than \( c_3 z_d e^{-c_4 A} \). (b) now follows by letting \( z_d \to 0 \).

A measure \( J(dx \, dy) \) on \( H(0) \times H(0) \) is symmetric if it remains unchanged under the map \( (x, y) \to (y, x) \).

Proposition 3.5. There exists a symmetric measure \( J \) such that
\[ \tilde{\mathcal{E}}(g, g) = \int_{H(0)} (g(x) - g(y))^2 J(dx \, dy). \quad (3.3) \]

Proof. Since the Dirichlet form \( \mathcal{E} \) for \( X \) is regular, with core \( C^2_0(\mathbb{R}^d) \), [FOT], Theorem 6.2.1 implies that \( \tilde{\mathcal{E}} \) is also regular, with core
\[ C' = \{ f : f = g|_H \text{ for some } g \in C^2_0(\mathbb{R}^d) \} = C^2_0(H(0)). \]
Hence, by [FOT], Theorem 3.2.1 \( \tilde{\mathcal{E}} \) can be written in the form \( \tilde{\mathcal{E}} = \tilde{\mathcal{E}}^{(c)} + \tilde{\mathcal{E}}^{(d)} + \tilde{\mathcal{E}}^{(k)} \), where
\[ \tilde{\mathcal{E}}^{(d)}(f, g) = \int \int (f(x) - f(y))(g(x) - g(y))J(dx \, dy); \]
here $J$ is a measure on $H(0) \times H(0)$ that is symmetric.

Since $Y$ is conservative, the killing term $\tilde{E}^{(k)} = 0$. By [JY], all martingales adapted to the filtration of $X$ can be written as stochastic integrals with respect to $d$ fixed martingales; the quadratic variation of each of these is absolutely continuous with respect to $dt$. Since $X$ spends zero time on $H(0)$, any continuous martingale on the filtration of $X$ which is constant except on \{ $t : X_t \in H(0)$ \} is therefore constant everywhere. It follows from [FOT], Section 5.3 that $\tilde{E}(c) = 0$.

When $f$ and $g$ are both $C_0^2$ with disjoint supports, we have using the symmetry of $J$

$$\tilde{E}(d)(f, g) = 2 \int \int f(x)g(y)J(dx\,dy). \quad (3.4)$$

Define a metric $d'$ on $H(0)$ by $d'(x, y) = d'((x', x_n), (y', y_n)) = |x' - y'|$. Since $|\nabla \gamma|$ is bounded, $d'$ is equivalent to the Euclidean metric.

**Theorem 3.6.** There exists a symmetric function $n(x, y)$, $x, y \in H(0)$ such that

$$\tilde{E}(g, g) = \int_{H(0)} (g(x) - g(y))^2 n(x, y) \nu(dx)\nu(dy). \quad (3.5)$$

The function $n(x, y)$ satisfies

$$c_1 |x - y|^{-d} \leq n(x, y) \leq c_2 |x - y|^{-d}, \quad d'(x, y) \leq 1, \quad (3.6)$$

$$\int_{d'(x, y) > \lambda} n(x, y) dy \leq c_2 e^{-\alpha \lambda}, \quad x \geq 1. \quad (3.7)$$

for constants $c_1, c_2, \alpha \in (0, \infty)$.

**Proof.** Let $f, g \in C_0^2(\mathbb{R}^d)$ with disjoint support. Choose a cube $D \subset \mathbb{R}^d$ which is large enough so that $f$ and $g$ are 0 outside $D$. Write $D_+ = D \cap I(0, \infty)$, $D_0 = D \cap H(0)$ and $D_- = D \cap I(-\infty, 0)$. Since $D_0$ has measure zero,

$$\tilde{E}(f, g) = \int_{\mathbb{R}^d} \nabla \Gamma f(x) \cdot \nabla \Gamma g(x) \sigma(x)^2 dx \quad (3.8)$$

$$= \int_{D_+} \nabla \Gamma f(x) \cdot \nabla \Gamma g(x) \sigma(x)^2 dx + \int_{D_-} \nabla \Gamma f(x) \cdot \nabla \Gamma g(x) \sigma(x)^2 dx.$$

By the Gauss-Green theorem, since $\mathcal{L} \Gamma g = 0$ on $H(0)^c$ and $f = 0$ on $\partial D - D_0$,

$$\int_{D_+} \nabla \Gamma f(x) \cdot \nabla \Gamma g(x) \sigma(x)^2 dx = \int_{D_0} \Gamma f(x) \sigma(x)^2 \frac{\partial \Gamma g(x)}{\partial n} dS,$$

where $dS$ is surface measure on $D_0$. But note that $\Gamma f = f$ on $H(0)$. 26
Using Proposition 3.4 and the fact that $f$ and $g$ have disjoint support, we may pass the limit under the integration sign to obtain

$$
\int_{D_+} \nabla \Gamma f(x) \cdot \nabla \Gamma g(x) \sigma(x)^2 dx = \int_{D_0} f(x) \sigma(x)^2 \int_{D_0} g(y) \frac{\partial \Gamma(\cdot, y)}{\partial n}(x) dS(y) dS(x).
$$

Summing with the analogous equality for the integral over $D_-$ and using Proposition 3.4, we obtain

$$
\tilde{\mathcal{E}}(f, g) = \int_{D_0} \int_{D_0} f(x) g(y) m(x, y) dS(y) dS(x).
$$

If we now compare this with (3.4), we see that $J(dx dy) = m(x, y) dS(x) dS(y)$. The symmetry of $J$ shows that $m$ is symmetric. Since $dS(x) \leq c_3 d\nu(x)$, the bounds in Proposition 3.4 complete the proof. \hfill \Box

Note that $\pi : H(0) \to \mathbb{R}^{d-1}$ is bijective. So, setting $Y_t = \pi(\tilde{X}_t)$, $n = d - 1$, $Y$ is a Markov process on $\mathbb{R}^n$, with Dirichlet form $\mathcal{E}_Y$ given by $\mathcal{E}_Y(f, f) = \tilde{\mathcal{E}}(f \circ \pi, f \circ \pi)$. Writing $n'(x', y') = n(\pi^{-1}(x'), \pi^{-1}(y'))$ we have

$$
\mathcal{E}_Y(f, f) = \iint (f(x') - f(y'))^2 n'(x', y') dx' dy'.
$$

It is immediate from Theorem 3.6 that $n'$ satisfies (1.1) and (1.2) so that $Y$ satisfies the hypotheses of Theorems 1.14 and 1.17.

Recall from the introduction that $h$ is $\mathcal{L}$-harmonic if $\mathcal{L}h = 0$.

**Theorem 3.7.** Let $\gamma$, $\sigma$ be as above, with $\sigma$ satisfying (S1)-(S3). Suppose $h$ is $\mathcal{L}$-harmonic, and $\sigma h$ is bounded. Then $h$ is constant.

**Proof.** By Proposition 2.7 $h$ is bounded. Set $M_t = h(X_t)$. Then, as in the proof of Proposition 2.7, $M$ is a martingale/$\tilde{\mathbb{P}}^x$ for any $x \in \mathbb{R}^d$. As $M$ is bounded, and $\tilde{\mathbb{P}}^x(\zeta_t < \infty)$ for all $t$, it follows that $h(\tilde{X}_{\zeta_t})$ is a martingale/$\tilde{\mathbb{P}}^x$. So, if $g$ is the function on $\mathbb{R}^n = \mathbb{R}^{d-1}$ defined by $g(x') = h(\pi^{-1}(x'))$, then $g$ is bounded and $g(Y_t)$ is a bounded martingale. Thus $g$ is $Y$-harmonic, and so $g$ is equal to a constant $c_0$ by Theorem 1.17. But then for any $x \in \mathbb{R}^d$, since $\tilde{\mathbb{P}}^x(\tau_0 < \infty) = 1$, and $h$ is bounded,

$$
h(x) = \tilde{\mathbb{E}}^x h(X_{\tau_0}) = c_0,
$$

which proves that $h$ is constant. \hfill \Box

4. Applications to PDE.

In this section, we apply the Liouville theorem (Theorem 3) to prove Theorems 1 and 2. First we have an elementary lemma.
**Lemma 4.1.** Let $\sigma = \partial u / \partial x_d$. Suppose $\sigma(0) > 0$ and for each $a$ there exists a constant $c(a)$ such that $a \cdot \nabla u(x) = c(a)\sigma(x)$ for all $x \in \mathbb{R}^d$. Then $u$ is of the form $u(x) = g(a \cdot x_d)$, for some fixed $a \in \mathbb{R}^d$ with $|a| = 1$.

**Proof.** Since $\sigma(0) > 0$ we have $\nabla u(0) \neq 0$; let $a$ be orthogonal to $\nabla u(0)$. Using the hypothesis shows $c(a) = 0$, so that $a \cdot \nabla u(x) = 0$ for all $x \in \mathbb{R}^d$, proving that $u$ is constant on every hyperplane orthogonal to $\nabla u(0)$.

**Proof of Theorem 1.** Let $\sigma(x) = \partial u(x) / \partial x_d$. It is shown in Lemma 3.2 of [GG], by using the moving plane method, that $\sigma(x) > 0$ in $\mathbb{R}^d$. Recall from the introduction that $\sigma$ satisfies

$$\Delta \sigma - F''(u(x))\sigma = 0, \quad x \in \mathbb{R}^d.$$ 

In view of Lemma 4.1 it is enough to prove that $\sigma$ satisfies the conditions (S1)-(S3) in Theorem 3.

We choose $\gamma(x') \equiv 0, x' \in \mathbb{R}^{d-1}$. Hence

$$I(a, b) = \{x = (x', x_d) \in \mathbb{R}^d : a \leq x_d \leq b, x' \in \mathbb{R}^{d-1}\}.$$ 

Since $F''(\pm 1) \geq \mu > 0$ and $u(x) \to \pm 1$ uniformly as $x_d \to \pm \infty$, we can choose $K_1$ large enough so that

$$\sigma^{-1}\Delta \sigma = F''(u(x)) > 2\varepsilon_1, \quad x \in I(-K_1, K_1)$$

for some $\varepsilon_1 > 0$. It can be shown, using (0.6), that $|u(x)| < 1, x \in \mathbb{R}^d$. The standard Schauder estimates for elliptic equations imply that $||\sigma||_\infty, ||\nabla \sigma||_\infty$ and $||\Delta \sigma||_\infty$ are all bounded by some $K_3 > 0$. Also, by [GT], $u$ and $\sigma$ are $C^{2+\varepsilon}$ in $\mathbb{R}^d$. Set $K_2 = 1 + K_1$; it remains to show that

$$\sigma \geq \varepsilon/2 > 0, \quad x \in I(-K_2, K_2)$$

for some positive constant $\varepsilon$.

If (4.1) is not true, there exists a sequence $\{x^{(m)}\} = \{(x^{(m)'} , x^{(m)}_d)\}$ such that $|x^{(m)}_d| \leq K_2$ and $\lim_{m \to \infty} \sigma(x^{(m)}) = 0$. Without loss of generality, we can assume $x^{(m)}_d \to x_d$ as $m \to \infty$. Now we define a sequence of solutions to (0.3)

$$u^{(m)}(x) := u(x^{(m)} + x), \quad x \in \mathbb{R}^d.$$ 

By the standard Schauder estimates for elliptic equations, we know that $\|u^{(m)}\|_{C^{3+\varepsilon}(\mathbb{R}^d)} \leq K_3 < \infty$. Therefore there exists a subsequence of $u^{(m)}$, which we still denote by $u^{(m)}$, and a solution of (0.3) $v(x) \in C^{3+\varepsilon}(\mathbb{R}^d)$ such that $\|u^{(m)} - v\|_{C^{2+\varepsilon}(\Omega)} \to 0$ as $m \to \infty$ for any bounded set $\Omega$. Since $|x^{(m)}_d| \leq K_2, v(x)$ converges uniformly to $\pm 1$ as $x_d$ tends to $\pm \infty$. So, by Lemma 3.2 of [GG],

$$\frac{\partial v}{\partial x_d} > 0, \quad x \in \mathbb{R}^d.$$ 

On the other hand, by the definition of $\{x^{(m)}\}$ we have

$$\frac{\partial u^{(0)}}{\partial x_d} = \lim_{m \to \infty} \frac{\partial u(x^{(m)})}{\partial x_d} = 0,$$
a contradiction. This proves (4.1), and so \( \sigma \) satisfies (S1)-(S3) with \( \varepsilon_0 = \min \{ \varepsilon_1, \varepsilon \} \). Theorem 1 now follows from Lemma 4.1 and the fact that the uniform convergence condition implies that the hyperplanes on which \( u \) is constant must be orthogonal to \( e^{(d)} \).

\[ \square \]

**Remark 4.2.** We can also prove directly by using suitable comparison functions that \( \sigma \) decays like \( \exp(\mu \pm x_d) \) near \( x_d = \pm \infty \) for some \( \mu \pm \), and hence that \( h \) is bounded. Then a weaker version of the Liouville theorem, not using Proposition 2.7, also leads to Theorem 1.

**Proof of Theorem 2.** We define \( \sigma, \psi, h \) as in the proof of Theorem 1. In order to show that \( \sigma \) satisfies conditions (S1)-(S3) in Theorem 3, we let \( \gamma(x'), x' \in \mathbb{R}^{d-1} \) be a level surface of \( u(x', x_d) \), say \( u(x', \gamma(x')) = 0, x' \in \mathbb{R}^{d-1} \). The function \( \gamma(x') \) is well defined in \( \mathbb{R}^{d-1} \) since \( u(x) \) is strictly monotone in \( x_d \), and \( \gamma \) is \( C^2 \) by the implicit function theorem. The cone condition implies that \( \vert \nabla \gamma(x') \vert \leq K_0 \) for \( x' \in \mathbb{R}^{d-1} \) for some \( K_0 < \infty \). Since \( F''(\pm 1) = \mu > 0 \), we can choose \( 0 < \delta < 1 \) and \( \varepsilon_1 > 0 \) such that \( F''(u) > 2\varepsilon_1 > 0 \) when \(-1 < u < -1 + \delta \) or \( 1 - \delta < u < 1 \). Let \( \gamma_1(x'), \gamma_2(x'), x' \in \mathbb{R}^{d-1} \) be the level surfaces of \( u(x) \) with \( u(x', \gamma_1(x')) = 1 - \delta, u(x', \gamma_2(x')) = -1 + \delta \) respectively. We claim that there exists \( \varepsilon_2 > 0 \) such that

\[
\sigma(x) > \varepsilon_2 / 2, \quad x \in \{x = (x', x_d) \in \mathbb{R}^d : \gamma_2(x') \leq x_d \leq \gamma_1(x'), x' \in \mathbb{R}^{d-1}\}. \tag{4.2}
\]

We prove this claim by contradiction. If it is not true, then there exists a sequence \( \{x^{(m)}\} = \{(x^{(m)'}, x_d^{(m)})\} \) such that \(-1 + \delta \leq u(x^{(m)}) \leq 1 - \delta \) and \( \lim_{m \to \infty} \sigma(x^{(m)}) = 0 \). As in the proof of Theorem 1, we define a sequence of solutions to (0.3)

\[
u^{(m)}(x) := u(x^{(m)}) + x, \quad x \in \mathbb{R}^d,
\]

and \( \|u^{(m)}\|_{C^{3+\varepsilon}(\mathbb{R}^d)} \leq K_3 < \infty \). Therefore, as before, there exists a subsequence of \( u^{(m)} \), which we still denote by \( u^{(m)} \), and a solution \( v(x) \in C^{3+\varepsilon}(\Omega) \) of (0.3) such that \( \|u^{(m)} - v\|_{C^{2+\varepsilon}(\Omega)} \to 0 \) as \( m \to \infty \) for any bounded set \( \Omega \).

Note that

\[
-1 + \delta \leq v(0) \leq 1 - \delta \tag{4.3}
\]

and

\[
\frac{\partial v(0)}{\partial x_d} = \lim_{m \to \infty} \frac{\partial u^{(m)}}{\partial x_d} = 0.
\]

Since \( \varphi = \frac{\partial v(x)}{\partial x_d} \geq 0, x \in \mathbb{R}^d \) satisfies

\[
-\Delta \varphi + F''(v(x)) \varphi = 0, \quad x \in \mathbb{R}^d,
\]

the strong maximum principle (see [GT]) yields \( \frac{\partial v(x)}{\partial x_d} \equiv 0, x \in \mathbb{R}^d \).

Since \( |v(0)| \leq 1 - \delta, v \) cannot be identically 1 or \(-1 \). It follows that \(-1 < v(x) < 1, x \in \mathbb{R}^d \). The Lipschitzian condition on \( u \) leads to the same condition on \( v \), i.e.

\[
|\nabla_x v(x)| \leq L(v) \frac{\partial v(x)}{\partial x_d}, \quad x \in \mathbb{R}^d.
\]
Therefore $\nabla v \equiv 0$, $x \in \mathbb{R}^d$. Hence $v(x)$ must be a constant, and so $v(x) \equiv u_0$, $x \in \mathbb{R}^d$, where $u_0$ is the unique critical point of $F$ in $(-1, 1)$. We now show that this is impossible.

For any ball $B_R(0) \subset \mathbb{R}^d$, we know that the first eigenvalue $\lambda_1 > 0$ and eigenfunction $\varphi_1(x) > 0$, $x \in B_R(0)$ of $-\Delta$ in the Sobolev space $H^1_0(B_R(0))$ satisfy

$$\begin{cases}
\Delta \varphi_1(x) + \lambda_1 \varphi_1(x) = 0, & x \in B_R(0), \\
\varphi_1(x) = 0, & x \in \partial B_R(0).
\end{cases}$$

Since $F''(u_0) < 0$, we can choose $R$ sufficiently large such that $\lambda_1 < -F''(u_0)/2$. On the other hand, when $m$ is large enough we have $-F''(u^{(m)}(x)) \geq -F''(u_0)/2$, $x \in B_R(0)$.

Since $\sigma^{(m)} := \frac{\partial u^{(m)}(x)}{\partial x_d} > 0$, $x \in B_R(0)$ satisfies

$$-\Delta \sigma^{(m)}(x) + F''(u^{(m)}(x))\sigma^{(m)}(x) = 0, \quad x \in B_R(0),$$

the quotient $\varphi_1(x)/\sigma^{(m)}(x) > 0$ satisfies

$$\Delta \varphi + 2 \frac{\nabla \sigma^{(m)}}{\sigma^{(m)}} \cdot \nabla \varphi + V(x) \varphi = 0, \quad x \in B_R(0)$$

where $V(x) = \lambda_1 + F''(u^{(m)}(x)) \leq 0$, $x \in B_R(0)$. This contradicts the maximum principle for (4.4) since $\varphi_1(x)/\sigma^{(m)}(x)$ vanishes on $\partial B_R(0)$. Therefore we have proven (4.2).

Since $u$ is bounded it follows immediately from (4.2) that there exists $K_1 < \infty$ such that

$$0 < \gamma_1(x) - \gamma(x') < K_1, \quad 0 < \gamma(x) - \gamma_2(x') < K_1, \quad x' \in \mathbb{R}^{d-1},$$

so that

$$\sigma^{-1} \Delta \sigma = F''(u) \geq 2\varepsilon_2 \quad \text{on} \quad I(-K_1, K_1)^c.$$

Let $K_2 = K_1 + 1$; it remains to show that for some $\varepsilon_0 > 0$

$$\sigma(x) > \varepsilon_0/2, \quad x \in I(-K_2, K_2).$$

We define $u^{(m)}$ and $v(x)$ as in the proof of (4.2). As before, we have $\frac{\partial v(0)}{\partial x_d} = 0$ and $\frac{\partial v(x)}{\partial x_d} \geq 0$, $x \in \mathbb{R}^d$. By the maximum principle (see [GT]) this implies that

$$\frac{\partial v(x)}{\partial x_d} \equiv 0, \quad x \in \mathbb{R}^d.$$  \hfill (4.7)

But from (4.5) and the definition of $\gamma_1, \gamma_2$, we have

$$v(0, x_d) > 1 - \delta, \quad \text{if} \quad x_d > 2K_1, \quad v(0, x_d) < -1 + \delta \quad \text{if} \quad x_d < -2K_1.$$  

This is a contradiction to (4.7), and therefore (4.6) holds.

So $\gamma$ and $\sigma$ satisfy the hypotheses of Theorem 3, and Theorem 2 now follows by Lemma 4.1. \hfill \Box
References


