Heat kernels and sets with fractal structure

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Contents

1. Fractal sets

In these notes I will concentrate mainly on heat kernel estimates on 'pre-fractal graphs' – for what these are see below. In the final section I will then explain briefly how these estimates can be used to define diffusion processes on some regular true fractal sets. For other (mathematical) surveys of this area see [**3**], [**49**], and Kigami's book [**33**]. A survey of the physics literature can be found in [**29**].

I begin by defining the geometrical objects we will be looking at. The following is an efficient way of defining a large class of fractals – see [**30**], [**43**]. Let $M \geq 2$, and $\psi_1 \dots \psi_M : \mathbb{R}^d \to \mathbb{R}^d$ satisfy

$$|\psi_i(x) - \psi_i(y)| = L^{-1}|x - y| \tag{1.1}$$

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for some $L > 1$. Following [**30**] we call the ψ_i *similitudes.* For $A \subset \mathbb{R}^d$ set

$$\Psi(A) = \bigcup_{i=1}^{M} \psi_i(A), \tag{1.2}$$

and let $\Psi^{(n)}$ denote the n-fold composition of Ψ.

DEFINITION. Let $\mathcal{K}$ be the set of non-empty compact subsets of $\mathbb{R}^d$. For $A \subset \mathbb{R}^d$ set $\delta_\varepsilon(A) = \{x : |x-a| \le \varepsilon \text{ for some } a \in A\}$. The *Hausdorff metric* ρ_H on $\mathcal{K}$ is defined by

$$\rho_H(A,B) = \inf\{\varepsilon > 0 : A \subset \delta_\varepsilon(B) \text{ and } B \subset \delta_\varepsilon(A)\}.$$

THEOREM 1.1 (See Chapter 9 of [**19**]). *(a)* ρ_H *is a metric on* $\mathcal{K}$.
(b) $(\mathcal{K}, \rho_H)$ *is complete.*
(c) If $\mathcal{K}_N = \{K \in \mathcal{K} : K \subset \overline{B}(0,N)\}$ *then* $\mathcal{K}_N$ *is compact in* $(\mathcal{K}, \rho_H)$.

It is not hard to verify that Ψ is a contraction in $(\mathcal{K}, \rho_H)$, and this leads to:

THEOREM 1.2. *([**30**],[**43**]). Let* $(\psi_1, \dots, \psi_M)$ *be as above. Then there exists a unique* $F = F_\Psi \in \mathcal{K}$ *such that* $F = \Psi(F)$. *Further, if* $G \in \mathcal{K}$ *then* $\Psi^n(G) \to F$ *in* $(\mathcal{K}, \rho_H)$. *If* $K_0 \in \mathcal{K}$ *satisfies* $\Psi(K_0) \subset K_0$ *then* $F = \cap_{n=0}^\infty \Psi^{(n)}(K_0)$.

EXAMPLES. 1. Sierpinski gasket in $\mathbb{R}^2$. Let $a_1 = (0,0)$, $a_2 = (1/2, \sqrt{3}/2)$, $a_3 = (1,0)$ be the three vertices of the unit triangle in $\mathbb{R}^2$, and

$$\psi_i(x) = a_i + \tfrac{1}{2}(x - a_i), \quad 1 \le i \le 3. \tag{1.3}$$

(So $M = 3$ and $L = 2$.) Then the fixed point $F = F_{\mathrm{SG}}$ is called the *Sierpinski gasket.* If we take K_0 to be the closed convex hull of the a_i then, writing $K_n = \Psi^{(n)}(K_0)$ we have $K_1 \subset K_0$ and $F = F_\Psi = F_{\mathrm{SG}} = \cap_{n=0}^\infty K_n$.

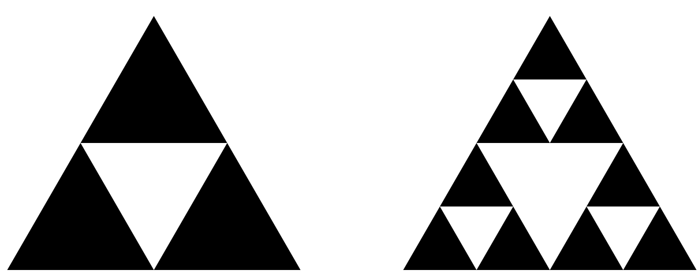

Figure 1.1. The first two stages in the construction of the Sierpinski gasket.

2. Sierpinski carpet in $\mathbb{R}^2$. Here we take $M = 8$, and $L = 3$. Write $K_0 = [0,1]^2$. Each ψ_i is a rotation-free map, so that $\psi_i(x) = a_i + \frac{1}{3}(x - a_i)$. The points a_i are the four corners of K_0 and the four mid-points of the sides of K_0.

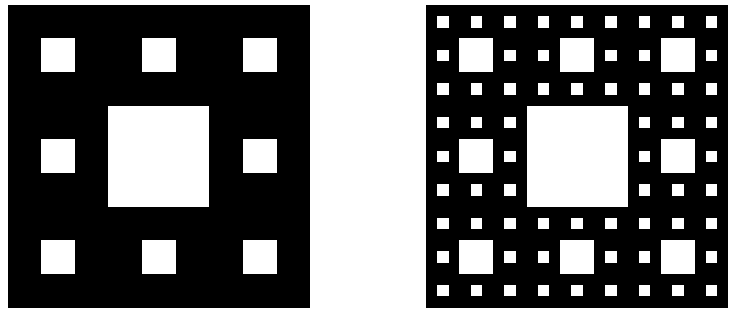

Figure 1.2. The sets K_2 and K_3 for the Sierpinski carpet

It is often easier to define the ψ_i by their action on K_0: with a bit of familiarity with how things work, given K_0, K_1, M and L it is easy to see what the ψ_i should be.

3. Generalised Sierpinski gasket in $\mathbb{R}^d$. The d-dimensional SG has $L = 2$ and $M = d+1$: the maps ψ_i are again given by (1.3) with the a_i being the vertices of the regular d-dimensional tetrahedron. We can form a larger family of sets (GSGs) by a similar method. For simplicity I just describe the procedure if $d = 2$. Choose $L \geq 2$, and divide the unit triangle K_0 into L^2 congruent triangles (some facing up, some down) of side L^{-1}. K_1 is chosen by removing all the downward triangles, and possibly some of the upward ones. We require that K_1 be connected, and have all the symmetries of K_0. The set of similitudes $\{\psi_i, 1 \leq i \leq M\}$ is then specified uniquely by requiring the ψ_i should be distinct, rotation free, and $\Psi(K_0) = K_1$.
4. Generalised Sierpinski carpets (GSCs) are defined in a similar fashion. Here we require that $\text{int}(K_1) \subset K_0 = [0,1]^d$ should be connected and have all the symmetries of the cube. (See [**9**] for details.)

One should note that while the 'iterated function system' $\{\psi_1, \ldots, \psi_M\}$ specifies F uniquely, the converse is not true.

This construction of F also gives a natural 'flat' measure $\mu = \mu_F$ on F. Define the 'word space' $\mathbb{W} = \{1, \ldots, M\}^{\mathbb{N}}$, and let μ_0 be the measure on $\mathbb{W}$ which gives each cylinder set

$$\mathbb{W}(i_1, \ldots, i_n) = \{w \in \mathbb{W} : w_j = i_j, 1 \leq j \leq n\},$$

measure M^{-n}. If $w \in \mathbb{W}$ then there exists a unique point $\pi(w) \in F$ with $\{\pi(w)\} = \cap_n \mathbb{W}(w_1, \ldots w_n)$. We then let $\mu(A) = \mu_0(\pi^{-1}(A))$ for measurable $A \subset F$.

We can calculate the dimension of the limiting set F from $(\psi_1, \ldots, \psi_M)$. However an "non-overlap" condition is necessary.

DEFINITION. $(\psi_1, \ldots, \psi_M)$ satisfies the *open set condition* if there exists an open set U such that $\psi_i(U)$, $1 \leq i \leq M$, are disjoint, and $\Psi(U) \subset U$.

Note that, since $\Psi(\overline{U}) \subset \overline{U}$, the fixed point F of Ψ satisfies $F = \cap \Psi^{(n)}(\overline{U})$.

For the Sierpinski gasket, if K_0 is the convex hull of $\{a_1, a_2, a_3\}$, then one can take $U = \text{int}(K_0)$.

THEOREM 1.3 (See [**19**], p. 119). *Let $(\psi_1, \dots, \psi_M)$ satisfy the open set condition, and let F be the fixed point of Ψ. Let*

$$\alpha = \frac{\log M}{\log L}. \tag{1.4}$$

Then $\dim_H(F) = \alpha$, *and* $0 < \mathcal{H}^\alpha(F) < \infty$.
(b) In addition we have

$$\mu(F \cap B(x,r)) \asymp r^\alpha, \quad x \in F,\ 0 \le r \le \operatorname{diam}(F). \tag{1.5}$$

Here, and elsewhere, we write $f(r) \asymp g(r)$ to mean that there exist constants $0 < c_1 \le c_2 < \infty$ such that $c_1 f(r) \le g(r) \le c_2 f(r)$.

REMARKS. 1. Following Lindstrøm [**39**] we call $L(= L_F = L_\Psi)$ the *length scaling factor of F* (or more stricly Ψ) and M the *volume or mass scaling factor.*
2. Both the basic SC and SG satisfy the condition that the boundary of K_0 is never removed. It follows from this that if we write $d_A(x,y)$ for the length of the shortest path in A between x and y then there exists a constant c_1 independent of n such that

$$|x - y| \le d_{K_n}(x,y) \le c_1 |x - y|. \tag{1.6}$$

This is not true for general GSGs or GSCs: consider for example the GSC, with $L = 5$ and $M = 12$ given by the following set K_1. In this case, if x_i are the corners of K_0 then one has $d_{K_n}(x_0, x_1) = (6/5)^n$.

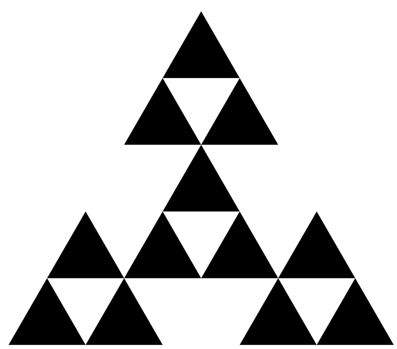

Figure 1.3. A generalised Sierpinski gasket.

ASSUME FURTHER:
1. The maps ψ are rotation-free, so that $\psi_i(x) = a_i + L^{-1}(x - a_i)$ for some a_i.
2. $a_1 = 0$.

We now define the *unbounded fractal*

$$\widetilde{F} = \widetilde{F}_{\rm SG} = \cup_{n=0}^\infty \psi_1^{-n}(F).$$

This creates an unbounded set made up of copies of the compact fractal F. Note that in fact $\psi_1^{-n}(F) \uparrow \widetilde{F}$, and that for each $R > 0$ there exists n_R such that $B(0,R) \cap \widetilde{F} = B(0,R) \cap \psi_1^{-n}(F)$ for all $n \ge n_R$.

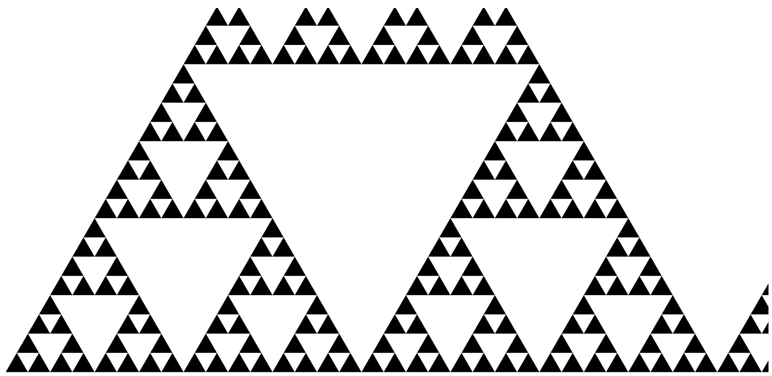

Figure 1.4. The unbounded Sierpinski gasket.

We will begin by looking at random walks on 'pre-fractal' graphs.

DEFINITION. Let (F_1, d_1), (F_2, d_2) be two metric spaces. A map $\varphi : F_1 \to F_2$ is a *rough isometry* if there exist constants A, B such that

$$A^{-1}(d_1(x,y) - B) \leq d_2(\varphi(x), \varphi(y)) \leq A(d_1(x,y) + B),$$

and for all $x_2 \in F_2$ there exists $x_1 \in F_1$ with $d_2(\varphi(x_1), x_2) \leq B$. If there exists a rough isometry between two spaces they are said to be *roughly isometric*. (One can check this is an equivalence relation.)

For example $\mathbb{Z}^d$ and $\mathbb{R}^d$ are roughly isometric: one rough isometry is the map $\varepsilon : \mathbb{Z}^d \to \mathbb{R}^d$ given by $\varepsilon(x) = x$.

DEFINITION. A graph $\Gamma = (G, E)$ is a *'pre-fractal' graph associated with* $\widetilde{F}_\Psi$ if G and $\widetilde{F}_\Psi$ are roughly isometric. One can define pre-fractal manifolds or domains in $\mathbb{R}^d$ in a similar way.

Although we will be ultimately able to say something about some general pre-fractal graphs, at this point the definition above is too wide.

DEFINITION. The *graphical Sierpinski gasket* Γ_{SG} is the graph with vertex set G consisting of corners of triangles side 1 in $\widetilde{F}$. $\{x, y\}$ is an edge if x and y are both corners of the same triangle of side 1.

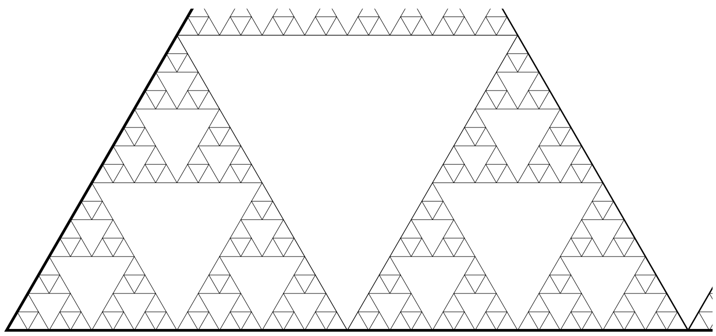

Figure 1.5. The graphical Sierpinski gasket.

We have therefore studied three classes of sets:

- 'Small sets' – i.e. compact true fractals.
- 'Big sets' – such as pre-fractal graphs.
- 'Small and big sets' – unbounded true fractals such as $\widetilde{F}$.

For the big sets there is a natural process X (the simple random walk on the graph, Brownian motion on the manifold), associated with the standard Laplacian Δ on the space. This is the graph Laplacian given by (2.1) below, in the graph case, or the Laplace-Beltrami operator in the case of a pre-fractal manifold. For both X and Δ the local properties are well understood, and the interesting questions concern the behaviour of the process at large times, and the global properties of harmonic or parabolic functions associated with Δ.

For the small sets the initial problem is the existence of a suitable Laplacian type operator $\mathcal{L}$, and the asociated Markov process Z. If these do exist then one can ask about their properties, both locally, for small time, and (in the 'small and big' case) globally and for large time.

Historically, after just enough work on the 'big' case to give what was needed, attention concentrated on the 'small' case. I will begin with the 'big' case.

Remark. In these notes I will concentrate on the families of generalised Sierpinski gaskets and carpets defined above. These families of spaces are big enough to give a wide family of examples of heat kernel behaviour. However, two additional classes of spaces have been very widely studied in the diffusions on fractals literature, and it may be useful to give a brief account of these.

Recall that a set F is *finitely ramified* if F can be disconnected by removing a finite number of points. Otherwise F is *infinitely ramified*. Generalised SGs are finitely ramified, since the part of F inside any small triangle can be disconnected from the rest of F by removing the three vertices of the triangle. It is easy to see that generalised SCs are infinitely ramified.

In the end there is no great difference in the behaviour of heat kernels on 'gasket' or 'carpet' type spaces. However, as far as proofs are concerned, finitely ramified sets are much easier to handle, and the two additional classes both consist of finitely ramified fractals.

Nested fractals were defined by Lindstrøm in [**39**]. These are subsets of $\mathbb{R}^d$ defined by M similitudes each with the same contraction factor L, and satisfying the open set condition. Additional hypotheses are:
(i) a strong symmetry condition,
(ii) a non-overlap condition, stronger than the open set condition, which forces the set F to be finitely ramified.

The results on heat kernels below all hold for nested fractals – see [**34**]. Examples of nested fractals are generalised SGs, Vicsek sets (see Figures 2.2 and 2.3 below) and the pretty 'Lindstrøm snowflake':

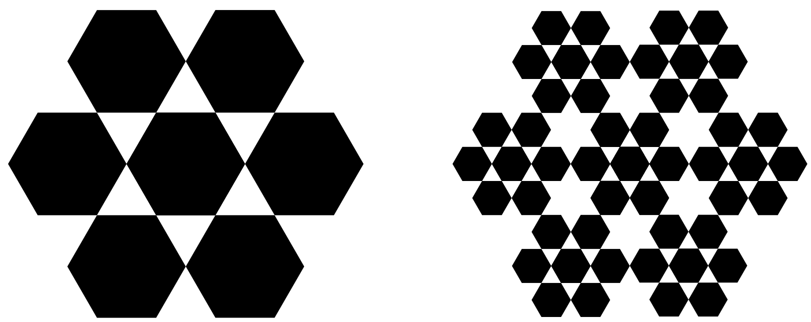

Figure 1.6. The Lindstrøm snowflake.

P.c.f. self-similar sets were introduced by Kigami in [**32**]: these are defined axiomatically as topological spaces, rather than as subsets of an space such as $\mathbb{R}^d$. Associated with such a set F are M maps $\psi_i : F \to F$ which play a similar role to the similitudes in the Moran-Hutchinson theory. 'p.c.f.' stands for 'post-critically finite': the post critical set is a subset $\mathcal{P}$ of the word space $\mathbb{W}$, and the condition that $\mathcal{P}$ is finite forces the sets $\psi_i(F)$ to overlap at only finitely many points, so that F is finitely ramified. There is no symmetry condition, and the absence of a canonical metric means that it does not make sense to ask if the sets $\psi_i(F)$ are all the same size. Nested fractals are a subset of this class.

See [**26**], [**27**] for work on heat kernels on these spaces. Even thought they are finitely ramified the situation is considerably more complicated than for generalised SGs or SCs, since there may be a number of different 'directions' in the space.

2. The graphical Sierpinski gasket

2.1. Graphs and random walks.

Graph notation. Let $\Gamma = (G, E)$ be an infinite connected graph. Write $x \sim y$ if $\{x, y\} \in E$ and set

$$a_{xy} = \begin{cases} 1 & \text{if } x \sim y, \\ 0 & \text{otherwise.} \end{cases}$$

(One may also consider weighted graphs, where the conductances a_{xy} are positive reals.) Let $\mu_0(x) = \sum_y a_{xy}$, and extend to a measure μ_0 on G. We assume that Γ

is *locally finite*, so that $\mu_0(x) < \infty$ for all x. Write $d(x,y)$ for the graph distance (the length of the shortest path connecting x and y), and let

$$B(x,r) = \{y : d(x,y) < r\}, \quad V(x,r) = \mu_0(B(x,r)).$$

The Laplacian on Γ is defined by

$$\Delta f(x) = \frac{1}{\mu_0(x)} \sum_y a_{xy}(f(y) - f(x)), \tag{2.1}$$

for $f \in C_0(G)$. Define also the Dirichlet form

$$\mathcal{E}(f,g) = \tfrac{1}{2} \sum_x \sum_y a_{xy}(f(x) - f(y))(g(x) - g(y)), \quad f, g \in L^2(G, \mu_0).$$

Then Δ is self-adjoint on $L^2(G, \mu_0)$ and we have the discrete version of the Gauss-Green formula (which is also the general relation between an Laplacian and the associated Dirichlet form):

$$(-\Delta f, g) = \mathcal{E}(f,g) = (f, -\Delta g).$$

The *simple random walk* on Γ is the Markov chain $X = (X_n, n \geq 0, \mathbb{P}^x, x \in G)$ with transition probabilities given by

$$\mathbb{P}^{\cdot}(X_{n+1} = y | X_n = x) = \frac{a_{xy}}{\mu_0(x)}.$$

The transition density (or heat kernel) of X with respect to μ_0 is

$$p_n(x,y) = \frac{\mathbb{P}^x(X_n = y)}{\mu_0(y)}.$$

This is easily seen to be symmetric, and satisfies the discrete heat equation on G:

$$p_{n+1}(x_0, x) - p_n(x_0, x) = \Delta_x p_n(x_0, x). \tag{2.2}$$

Closely related is the *continuous time simple random walk*, $Y = (Y_t, t \geq 0)$, on Γ. This is the continuous time Markov process on G with generator Δ. Y waits for a negative exponential time at each vertex x, and then jumps to one of the neighbouring vertices in the same way as X. We write

$$q_t(x,y) = \frac{\mathbb{P}^x(Y_t = y)}{\mu_0(y)}$$

for the heat kernel for Y. This is a smoother object than p_n, and is slightly easier to handle; it satisfies the equation

$$\frac{\partial q_t(x_0, x)}{\partial t} = \Delta_x q_t(x_0, x). \tag{2.3}$$

Estimates for p_n transfer easily to q_t, since one has

$$q_t(x,y) = \sum_{n=0}^{\infty} \frac{e^{-t} t^n}{n!} p_n(x,y). \tag{2.4}$$

2.2. Simple random walk on the graphical SG. We begin by calculating the volume growth of $\Gamma = \Gamma_{\rm SG}$. If A is any triangle side 2^n then $\mu_0(A) \simeq 2.3^n$, and any point in A is a distance of 2^n or less from each of the corners. If $r > 1$ choose $n \geq 0$ so that $2^n \leq r < 2^{n+1}$. Then $B(x,r)$ contains a triangle in Γ of side 2^n, and is contained in 4 triangles of side 2^{n+1}. Thus $c3^n \leq V(x,r) \leq c'3^n$, so if we set

$$\alpha = \alpha_{\rm SG} = \frac{\log 3}{\log 2} = \frac{\log M}{\log L} \tag{2.5}$$

then, as $3^n = 2^{n\alpha}$, we have

$$cr^\alpha \leq V(x,r) \leq c' r^\alpha. \tag{V_α}$$

This condition easily implies volume doubling:

$$V(x,2R) \leq cV(x,R), \quad x \in G,\ R \geq 1. \tag{VD}$$

The key to the SRW on the graphical SG is to calculate the crossing times of 'triangles'. Consider a SRW X on G started at 0. Set

$$T_n = \min\{r \geq 0 : |X_r| = 2^n\};$$

clearly X_{T_n} is at one of the two vertices of the triangle side 2^n whose third vertex is 0.

LEMMA 2.1. *$\mathbb{E}^0 T_0 = 1$ and $\mathbb{E}^0 T_1 = 5$.*

PROOF. The first statement is obvious, and the second is a simple calculation. Label the vertices as in Figure 2.1, and write $h(x) = \mathbb{E}^x T_1$.

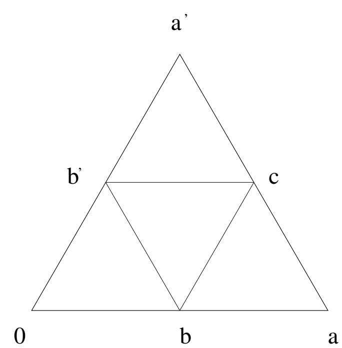

Figure 2.1. The neighbourhood of 0 in $\Gamma_{\rm SG}$.

Then $h(x) = h(x')$ by symmetry, $h(a) = h(a') = 0$ and,

$$\begin{aligned} h(0) &= 1 + \tfrac{1}{2}(h(b) + h(b')) \\ h(b) &= 1 + \tfrac{1}{4}(h(0) + h(b') + h(a) + h(c)) \\ h(c) &= 1 + \tfrac{1}{4}(h(b) + h(b') + h(a) + h(a')); \end{aligned}$$

from which it follows that $h(0) = 5$. □

REMARK. We will see that the numbers 2, 3 and 5 play a key role in analysis on the SG and graphical SG. The first two of these are just L and M, the length and volume scaling factors. These numbers are in a sense 'geometric', since they can be calculated simply from the geometry of the SG. The third number, 5, is in some (rather unclear) sense deeper – it seems that it can only be obtained via some analytic calculation. This number is called by Lindstrøm the *time scaling factor*, denoted $T = T_F$. The calculation above using hitting times is one route to finding T_F, but there are others, as we will see below.

For Sierpinski carpets L_F and M_F are easily calculated, but T_F is not known exactly.

THEOREM 2.2. $\mathbb{E}^0 T_n = 5^n$.

PROOF. This is easy to prove using a very strong and nice exact scaling property of G and X. Define

$$G^{(n)} = \{x \in G : 2^{-n}x \in G\}.$$

These are the vertices of triangles of side 2^n. Fix n and define (stopping) times

$$S_0 = 0, \qquad S_{r+1} = \min\{k \geq S_r : X_k \in G^{(n)} - \{X_{S_r}\}\}, \quad r \geq 1.$$

Using symmetry (and the fact that we can reflect the r.w. running in the two n-triangles containing X_{S_r} into just one n-triangle) we have that $\mathbb{E}^0(S_{r+1} - S_r) = \mathbb{E}^0 T_n$.

Now let

$$V_r = 2^{-n} X_{S_r}.$$

A little thought shows that $(X_{S_r}, r \geq 0)$ is a random walk on $G^{(n)}$, and hence V is a SRW on G, and so has the same law as X. At the time k that $|X_k|$ first equals 2^{n+1} we will have $k = S_N$ for some (random) N, with the same probability distribution as T_1. So, since $S_r - S_{r-1}$ and N are independent,

$$\mathbb{E}^0 T_{n+1} = \mathbb{E}^0 S_N = \mathbb{E}^0 \sum_{r=0}^{N} (S_r - S_{r-1}) = \mathbb{E}^0(N)\mathbb{E}^0 T_n = 5\mathbb{E}^0 T_n.$$

Iterating gives $\mathbb{E}^0 T_n = 5^n$. □

REMARKS. 1. Using the same ideas one can show that the crossing times of triangles give rise to a branching process. In particular one can also obtain estimates of higher moments and tails of the crossing times T_n. This was the basis of the estimates in [**14**] for heat kernels on the Sierpinski gasket.
2. Exactly the same argument works for generalised SGs, but of course with different values of L_F, M_F, T_F.
3. This property of the random walk on the SG is called 'decimation invariance'. As we have seen above it works very well for the SG, so well that it is in fact potentially misleading:

- It can be hard to find (or prove that there exists) a decimation invariant random walk on a fractal,
- Even though decimation invariant random walks do not exist on fractals such as the Sierpinski carpet one can nevertheless say a fair amount about these spaces.

Now define for a fractal F

$$\beta = \beta_F = \frac{\log T_F}{\log L_F}.$$

In particular $\beta_{\rm SG} = \log 5/\log 2 \simeq 2.32$. Set

$$\tau(x,r) = \inf\{n : X_n \notin B(x,r)\},$$

and say that Γ satisfies the condition E_β if

$$(E_\beta) \qquad \mathbb{E}^x \tau(x,r) \asymp r^\beta, \quad x \in G,\ r \geq 1.$$

COROLLARY 2.3. *(a) The SRW on $\Gamma_{\rm SG}$ satisfies $(E_{\beta_{\rm SG}})$*
(b) For $x \in G$, $t \geq 1$,

$$\mathbb{E}^x d(x, X_t)^2 \asymp t^{2/\beta_{\rm SG}}. \tag{2.6}$$

PROOF. (a) is proved by approximating balls from within and without by triangles. (b) can then be proved using (a) and bounds on the tails of T_n, but it follows easily from the bounds on the transition probabilities of X given in Theorem 2.4. □

REMARKS. 1. The property (2.6) is called 'subdiffusive' behaviour in the mathematical physics literature. (See for example [**1**], [**29**], [**46**]). The meaning is clear – the diffusion (or random walk: physicists see little difference) moves on average more slowly than a diffusion in Euclidean space. More generally 'anomalous diffusion' is used to refer to situations where either subdiffusive or superdiffusive behaviour occurs. See [**50**] for an account of how anomalous diffusion can arise from homogenization on different length scales.
2. The calculations above were historically a preparation (in [**36**], [**21**], [**14**]) to proving the existence of a continuum limit. I will return to this below – but given the decimation invariant structure and the special roles of 2 and 5 one may reasonably (and correctly) expect that the processes

$$Z_t^{(n)} = 2^{-n} X_{[5^n t]}, \quad t \geq 0,$$

will have a non-trivial and interesting limit Z.
3. To see the difficulties that one may encounter in even slightly more general situations, consider the Vicsek sets. These are obtained by a similar construction to the SG, but based on squares rather than triangles.

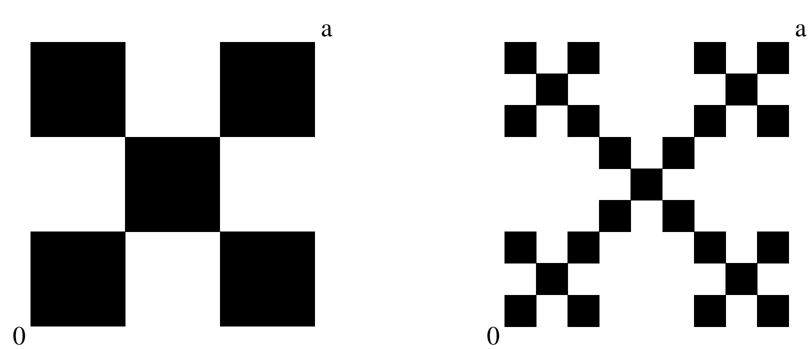

Figure 2.2. The Vicsek set with $L = 3$.

Consider first the case $L = 3$. The SRW on V which makes just horizontal and vertical jumps is not decimation invariant, since starting at 0 it can with positive probability move diagonally to the point a. If one allows diagonal transitions then it is quite easy to see that a diagonal transition probabilty of $p_D = \frac{1}{3}$ gives rise to a decimation invariant walk.

If $L = 5$ then a decimation invariant random walk still exists, and makes diagonal transitions with probability p_D, where p_D is the root of a polynomial equation. In both these cases $p_D = 1$ also gives rise to a decimation invariant random walk, but one which is degenerate in the sense that that it lives on a strict subset of G.

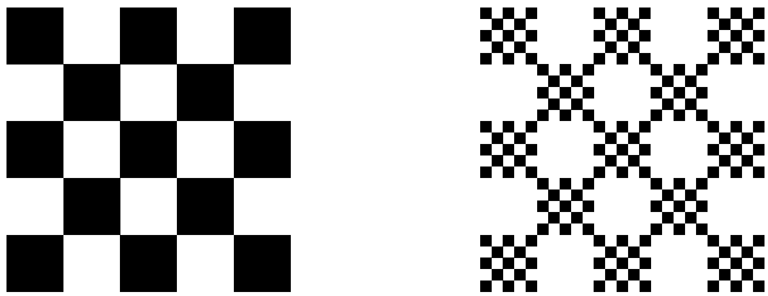

Figure 2.3. A Vicsek set with $L = 5$.

The general question of what properties are needed for existence or uniqueness of a non-degenerate decimation invariant random walk is still open. In the p.c.f. case neither existence nor uniqueness holds in general – see [**28**] for non-existence, and [**42**] for non-uniqueness. For nested fractals Lindstrøm [**39**] proved existence, while Sabot [**47**] proved uniqueness, and also made substantial progress on the gerneral problem.

The heat kernel $p_n(x,y)$ satisfies the following bounds.

THEOREM 2.4. *Let $\Gamma = (G,E)$ be a graphical generalized SG or SC. There exist $\alpha \geq 1$ and $2 \leq \beta \leq 1+\alpha$ such that for $n \geq 1 \vee d(x,y)$*

$$\text{(HK}(\beta)) \qquad \begin{aligned} p_n(x,y) &\leq \frac{c_1}{V(x,n^{1/\beta})} \exp\Big(-c_2\Big(\frac{d(x,y)^\beta}{n}\Big)^{1/(\beta-1)}\Big). \\ p_n(x,y) + p_{n-1}(x,y) &\geq \frac{c_2}{V(x,n^{1/\beta})} \exp\Big(-c_1\Big(\frac{d(x,y)^\beta}{n}\Big)^{1/(\beta-1)}\Big). \end{aligned}$$

Thus the two numbers α and β capture much of the behaviour of the random walk. The first, α, is just the volume growth of the graph, while the second (which equals 2 for graphs such as $\mathbb{Z}^d$) gives the space-time scaling.

REMARKS. 1. In the diffusions on fractals literature (following the notation of [**1, 46**]) one writes $\alpha = d_f$ (for 'fractal dimension') and $\beta = d_w$ (for 'walk dimension'). The explanation for this term is that if $\beta < \alpha$, (and so there is room), one finds in the diffusion limit that the range of Z, $R(Z) = \{Z_t, t \geq 0\}$ is a set of dimension β.
2. It is not hard to see that the conditions (V_α) and (E_β) imply that $\alpha \geq 1$ and $2 \leq \beta \leq 1+\alpha$. In fact this is the only restriction on α and β – see [**4**].
3. Since $V(x,r) \asymp r^\alpha$ the term $V(x,n^{1/\beta})^{-1}$ can be replaced by $n^{-\alpha/\beta}$. Thus one has that X is recurrent when $\alpha \leq \beta$, and transient if $\alpha > \beta$.
4. Giving the lower bound for $p_n(x,y) + p_{n-1}(x,y)$ avoids parity problems.
5. Similar bounds hold for $q_t(x,y)$, the density of the continuous time random

walk.
6. It turns out (see [**3**], Proposition 3.42), that any finitely ramified fractal which satisfies HK(β) must have $\alpha < \beta$. So all GSGs are recurrent.

I will sketch the proof of this theorem below.

2.3. Analysis on the Sierpinski gasket.

DEFINITION. Let $\Gamma = (G, E)$ be a connected graph. For $A \subset G$ set $\partial A = \{y : y \sim x, \text{ for some } x \in A\}$, and $\overline{A} = A \cup \partial A$. A function $h : \overline{A} \to \mathbb{R}$ is *harmonic* on A if

$$\Delta h(x) = 0, \quad x \in A.$$

An *elliptic Harnack inequality (EHI)* holds for Γ if there exists $c_\Gamma < \infty$ such that if $x \in \Gamma$, $R \geq 1$ and $h \geq 0$ is harmonic in $B^* = B(x, 2R)$ then,

$$\text{(EHI)} \qquad \sup_{B(x,R)} h \leq c_\Gamma \inf_{B(x,R)} h.$$

If Γ is locally bounded, then one always has a local Harnack inequality, i.e. for $R = 2$. The point of EHI is that the constant c_Γ is independent of R.

If EHI holds for a graph Γ it has many consequences in terms of regularity of harmonic functions on Γ. One example is the following Liouville property.

LEMMA 2.5. *Let Γ be a graph satisfying (EHI). If h is a positive harmonic function on the whole of Γ then h is constant.*

PROOF. We can assume $h \geq 1$ everywhere, and that there exists x_0 with $h(x_0) < 2$. Applying EHI in $B = B(x_0, R) \subset B(x_0, 2R)$ we deduce that $\sup_B h \leq 2c_1$, so that h is bounded.

In addition EHI implies an oscillation inequality. Let $O_B(h) = \sup_B h - \inf_B h$. Replacing h by $h - b$ we can assume $\inf_{\overline{B}^*} h = 0$, and then the EHI implies that $O_B(h) \leq (1 - c_1^{-1}) O_{\overline{B}^*}(h)$. Iterating one deduces that if $B_n = B(x_0, 2^n)$ then $O_B(h) \leq (1 - c_1^{-1})^n ||h||_\infty$, from which the Liouville property is immediate. □

THEOREM 2.6. *EHI holds for Γ_{SG}.*

PROOF. We prove it for triangles, and an easy covering argument then gives it for balls.

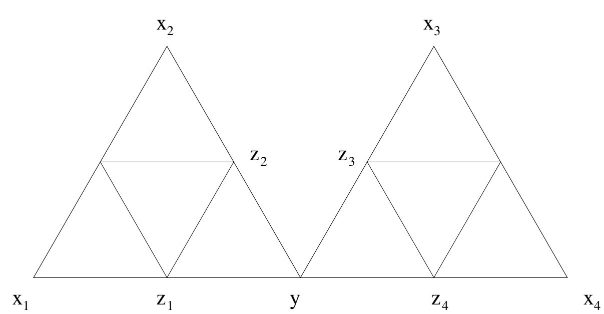

Figure 2.4. A region in Γ_{SG}.

A typical situation is given in Figure 2.4, where $|y - z_i| = 2^n$ and smaller detail is omitted. Let $A = B(y, 2^n)$, $A^* = B(y, 2^{n+1} - 1)$, and $h \geq 0$ be harmonic on A^*. The mean value property of harmonic functions implies that h attains its extremal values on $\overline{A}$ at the boundary points z_i. Using the decimation invariance of X (or the optional sampling theorem for the martingale $h(X)$) one can show that

$$h(y) = \tfrac{1}{4}\sum_{i=1}^{4} h(z_i),$$

with similar relations at each z_i. So for any i, j, $h(z_i) \geq \frac{1}{4}h(y) \geq (\frac{1}{4})^2 h(z_j)$, proving that $\sup_A h \leq 16 \inf_A h$. □

Now we recall the *parabolic Harnack inequality (PHI)* on a graph.

DEFINITION. We say $u = u(n,x)$ is *parabolic* on $[0,T]\times B^*$ (strictly, $([0,T]\cap\mathbb{Z})\times B^*$) if

$$u(n+1,x) - u(n,x) = \Delta u(n,x), \quad 0 \leq n < T, \quad x \in B^*.$$

The PHI holds for Γ if, writing $T = R^2$, whenever $u \geq 0$ on $[0,4T]\times\overline{B}^*$ is parabolic in $[0,4T]\times B^*$, writing $Q_- = [T,2T]\times B$, $Q_+ = [3T,4T)\times B$, and $u'(n,x) = u(n-1,x)+u(n,x)$ then (for a constant C_Γ depending only on Γ)

$$\text{(PHI(2))} \qquad \max_{Q_-} u \leq c_1 \min_{Q_+} u'.$$

(One has to replace u by u' to avoid parity problems.)

If h is harmonic on $\overline{B}^*$ then $u(n,x) = h(x)$ is parabolic, so PHI(2) implies EHI. The question of whether the converse was true was for some time an open question. However, it is easy, using Theorem 2.4, to see that PHI(2) fails for $\Gamma_{\rm SG}$. Let $u(n,x) = p_n(x_0,x)$, which is parabolic in $[0,\infty)\times G$. Then (neglecting constants) if x_1 is chosen with $d(x_0,x_1) = R-1$,

$$\max_{Q_-} u \simeq p_{R^2}(x_0,x_0) \simeq (R^2)^{-\alpha/\beta} = R^{-2\alpha/\beta},$$

$$\min_{Q_+} u \simeq p_{3R^2}(x_0,x) \simeq (3R^2)^{-\alpha/\beta} e^{-(R^\beta/3R^2)^{1/(\beta-1)}} \simeq R^{-2\alpha/\beta} e^{-R^{(\beta-2)/(\beta-1)}},$$

so that PHI(2) fails.

It is fairly clear why this example works: the EHI contains no information on the space-time scaling of the random walk, while PHI(2) does. We can extend the definition of PHI(2) to a family of parabolic Harnack inequalities PHI(β), for $\beta \geq 2$, by setting $T = R^\beta$ in the definition above.

PROPOSITION 2.7. *(*[**24**]*)* $\Gamma_{\rm SG}$ *satisfies PHI(*$\beta_{\rm SG}$*).*

This will follow from Theorem 2.11 below.

A well known result ([**22**], [**48**]) is that for manifolds PHI(2) is equivalent to volume doubling (VD) plus a Poincaré inequality, PI(2). This has been extended to graphs. We say $\Gamma = (G,E)$ satisfies the p_0 *condition* if there exists $p_0 > 0$ such that whenever $x \sim y$ then

$$\mathbb{P}^x(X_1 = y) \geq p_0.$$

(This immediately implies that x can have at most p_0^{-1} neighbours.)

THEOREM 2.8 (See [**17**], [**24**]). *The following are equivalent for a graph* $\Gamma = (G, E)$ *satisfying the* p_0 *condition.*
(a) Γ *satisfies PHI(2).*
(b) Γ *satisfies PI(2) and VD.*
(c) Γ *satisfies HK(2).*

It follows from this that $\Gamma_{\rm SG}$ also fails the Poincaré inequality PI(2).

To investigate further the Poincaré inequality on the graphical SG, let A_n be a triangle of side 2^n, and look for the best constant P_n in the inequality

$$\sum_{x\in A_n} (f(x) - \overline{f}_A)^2 \mu_0(x) \le P_n \sum_{x,y\in A_n} a_{xy}(f(x) - f(y))^2. \tag{2.7}$$

Here $\overline{f}_A = \mu_0(A_n)^{-1} \sum_{x\in A_n} f(x)\mu_0(x)$. Let $E(A_n)$ be the set of edges in A_n. If we define $|\nabla f| = |f(x) - f(y)|$ on an edge $\{x, y\}$, and extend a to a measure on the edges E then we can write (2.7) as

$$\int_{A_n} (f - \overline{f}_A)^2 d\mu_0 \le P_n \int_{E(A_n)} |\nabla f|^2 da = P_n \mathcal{E}_{A_n}(f, f).$$

(P_n^{-1} is the spectral gap for A_n.) I do not know the constants P_n explicitly. To estimate them, it is instructive to look at an easier problem, which is to find the 'effective resistance' across A_n.

Given a graph $\Gamma = (G, E)$ and disjoint subsets $B_i \subset G$, $i = 0, 1$ define the *effective resistance between* B_0 *and* B_1 by

$$R_{\rm eff}(B_1, B_2)^{-1} = \inf\{\mathcal{E}(f, f) : f = 0 \text{ on } B_0 \text{ and } f = 1 \text{ on } B_1\}.$$

The inverse of $R_{\rm eff}$ is the *effective conductance* and is closely connected to capacity.

For $\Gamma_{\rm SG}$ it is easy to compute effective resistances exactly, if we use a trick of the electrical engineers, the ∇-Y transform. This implies that any triangle (∇) of unit resistors can be replaced by a Y of wires of resistance $\frac{1}{3}$, without affecting the overall behaviour of the elctrical system. (See Section 4 of [**3**] for more details and mathematical background.)

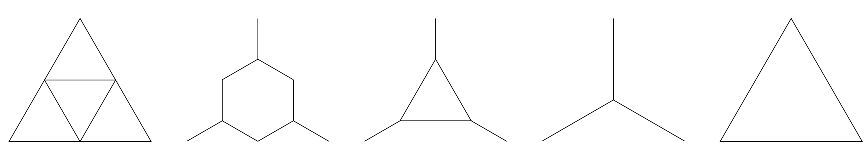

Figure 2.5. A sequence of ∇-Y transforms for the Sierpinski gasket.

Using this we see that (with inputs just at the edges) A_2 is electrically equivalent to a triangle with resistances $(5/3)^2$ on each edge. Iterating it follows that A_n is electrically equivalent to a triangle with resistances $(5/3)^n$ on each edge. So we can find a function f_n on A_n with $\int_{A_n} (f - \overline{f}_A)^2 \asymp 3^n$ and $\mathcal{E}_{A_n}(f, f) \asymp (3/5)^n$, which implies that $P_n \ge c5^n$.

It is no surprise that 5^n is actually the correct size:

THEOREM 2.9. *$\Gamma_{\rm SG}$ satisfies the Poincaré inequality PI(β) with $\beta = \beta_{\rm SG}$:*

$$(PI(\beta)) \qquad \int_{B(x,R)} (f - \overline{f})^2 \le cR^\beta \int_{B(x,R)} |\nabla f|^2.$$

REMARKS. 1. This can be derived from Theorem 2.4 by a standard argument – see [**48**]. (The anomalous diffusion case, for Sierpinski carpets, is given in [**8**], but the proof is essentially the same as in the $\beta = 2$ case.)
2. The PI can be viewed as arising from an optimisation problem

$$P_n = \max\{\int_{A_n} f^2 : \int_{A_n} f = 0, \, \mathcal{E}_{A_n}(f,f) = 1\}$$

in which one aims to maximise the L^2 norm of a function subject to control of its energy. A uniformly increasing f is clearly not the best choice: it is better to make $|\nabla f|$ large near the 'bottlenecks' in the SG, since this increases $\int f^2$ without a large penalty in terms of energy.

There is a close connection between random walks and graphs and effective resistance of electrical circuits – see for example [**18**], [**51**], and [**3**], Section 4. The following result on mean hitting times and resistance was first proved in [**16**]. A more accessible reference is in [**52**], but the proof is quite easy.

THEOREM 2.10. *Let $\Gamma = (G, E)$ be a finite graph, $x, y \in G$. Let $T_z = \min\{n \ge 0 : X_n = z\}$. Then*

$$\mathbb{E}^x T_y + \mathbb{E}^y T_x = R_{\rm eff}(x,y)\mu_0(G).$$

EXAMPLE. We can use this to verify the results of the previous resistance calculations on the SG. Let A_n be a triangle of side 2^n, and label the three edge vertices x_i, $0 \le i \le 2$. Let $T' = T_{x_1} \wedge T_{x_2}$. We saw previously that $\mathbb{E}^{x_0} T' = 5^n$.

To use Theorem 2.10 we also will need the symmetry of A_n. If $b = \mathbb{E}^{x_0} T_{x_1}$ then

$$b = \mathbb{E}^{x_0}(T_{x_1} - T') + \mathbb{E}^x T' = \tfrac{1}{2}\mathbb{E}^{x_2} T_{x_1} + 5^n = \tfrac{1}{2} b + 5^n.$$

So $b = 2.5^n$ and as $\mu_0(A_n) = 2.3^{n+1}$ we deduce that

$$R_{\rm eff}(x_0, x_1) = \frac{2\mathbb{E}^{x_0} T_{x_1}}{2.3^{n+1}} = \tfrac{2}{3}(\tfrac{5}{3})^n.$$

We remark that the effective resistance calculations we made earlier correspond exactly to the decimation of random walks. This follows from the 'Trace Theorem' of Fukushima – see [**20**], Theorem 6.2.1.

The simplest proof of Theorem 2.4 is to use following result.

THEOREM 2.11. [**24**]. *The following are equivalent for an infinite graph $\Gamma = (G, E)$ which satisfies the p_0 condition:*
(a) VD+EHI+(E_β),
(b) HK(β),
(c) PHI(β).

We have been able to prove the conditions in (a) quite easily for $\Gamma_{\rm SG}$, so using Theorem 2.11 we immediately obtain Theorem 2.4. I will discuss the proof of Theorem 2.11 below, but will first look at another family of examples.

3. Sierpinski carpets

3.1. Basic properties. As is mentioned above, the Sierpinski gasket is rather exceptional. The other class of fractals I will discuss, Sierpinski carpets still have some special features (symmetry in particular) but are general enough so that one needs to develop more powerful techniques to handle them. For a more detailed survey see [**15**].

The basic Sierpinski carpet F in $\mathbb{R}^d$ is the compact subset of $K_0 = [0,1]^d$ obtained by taking $3^d - 1$ (rotation free) similitudes with contraction factor $L = 3$, such that

$$K_1 = \Psi(K_0) = \cup_i \psi_i(K_0) = K_0 - (1/3, 2/3)^d.$$

We can take $\psi_1(x) = \frac{1}{3}x$. We set $K_n = \Psi^n(K_0)$, and $F = \cap_n K_n$. Let $M_d = 3^d - 1$ be the volume scaling factor; note that K_n consists of M_d^n cubes of side 3^{-n}. We also define two unbounded sets based on the SC:

$$\widetilde{F} = \cup_{n=0}^{\infty} \psi_1^{-n}(F), \qquad \widetilde{F}_0 = \cup_{n=0}^{\infty} \psi_1^{-n}(K_n).$$

The first, $\widetilde{F}$, is the unbounded SC, while the second is called the 'pre-SC'. Note that $\widetilde{F}_0$ is a union of cubes of side 1.

We also define the graphical Sierpinski carpet, $\Gamma_{\mathrm{SC}} = (G, E)$ by placing a vertices in the centre, and also in the centre of each face, of each cube of side 1. $\{x, y\}$ is edge if x is the centre of a cube C of side 1, and $y \in \partial C$.

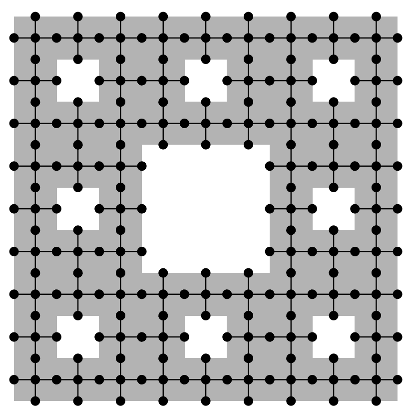

Figure 3.1. The graphical Sierpinski carpet

As in the case of the SG the basic geometry of the SC is quite easy, and by comparing balls with cubes it is straightforward to verify that Γ_{SC} satisfies (V_α) with

$$\alpha = \alpha_d = \alpha_{SC(d)} = \frac{\log M_d}{\log L} = \frac{\log(3^d - 1)}{\log 3}.$$

However, unlike the SG, there is no decimation invariance to make calculations of crossing times simple. The SC was studied in series of papers by Bass and myself [**5**]–[**10**], and proved quite a hard problem. In the end it is possible to prove most (but not all) of what can be proved for the SG.

In this account I will not follow the historical order of development, but will outline what is now seems the fastest route. This is:

(1) Prove an EHI for $\Gamma_{\rm SC}$ by probabilistic means.
(2) Use the connection with resistance to define $\beta = \beta_{SC}$.
(3) Prove that $\Gamma_{\rm SC}$ satisfies (E_β) using (1) and (2).
(4) Deduce $HK(\beta)$ from Theorem 2.11.
(5) Use $HK(\beta)$ to define a continuous process on $\widetilde{F}$.

I will just discuss the basic d-dimensional Sierpinski carpet, but the same arguments work for generalized SCs.

THEOREM 3.1. $\Gamma_{\rm SC}$ *satisfies (EHI).*

PROOF. This proof is unfortunately quite long, and uses the symmetry very strongly. The basic strategy is to use a 'coupling' argument, and to prove that if x_1, x_2 are points in a cube Q_n of side 3^n, both a distance (say) at least 3^{n-2} from Q_N^c, then one can define two SRW $(X_n^i, n \geq 0)$, $i = 1, 2$ on $\Gamma_{\rm SC}$. These satisfy $X_0^i = x_i$, and if

$$T_C = \min\{r : X_r^x = X_r^y\}, \quad \tau_i = \min\{r : X_r^i \notin Q_n\},$$

then one has for a constant p independent of n

$$P(T_C < \tau_1 \wedge \tau_2) \geq p > 0. \tag{3.1}$$

Note that X^1 and X^2 are not independent.

For random walks in a large cube one could prove (3.1) by 'reflection coupling' – see [**40**]. To prove (3.1) for $\Gamma_{\rm SC}$ the essential idea is that of 'coupling at level k' (where $k \in \{1, \ldots n\}$). This holds at time r there exists an isometry ϕ between the cubes of side 3^k containing X_r^1 and X_r^2 such that $\phi(X_r^1) = X_r^2$.

Once such a coupling at level k holds, at a time T_k, it is possible to maintain it, so that X_r^i remain coupled at level k for $r \geq T_k$. Using the symmetries of cubes one can show that, starting with random walks coupled at level k, one can couple at level $k+1$ before the random walks have moved more than $O(3^k)$. Finally, coupling at level n is (nearly) equivalent to $X_r^1 = X_r^2$.

If h is a harmonic function on Q_n then (3.1) implies an oscillation inequality. The EHI then follows from this, and a fairly easy estimate on the probability of hitting small balls, by an argument due to Cafarelli. For the details see Sections 3 and 4 of [**9**]. □

3.2. Resistance and crossing times. Let $\widetilde{F}_n = \widetilde{F}_0 \cap [0, 3^n]^d$ be a cube in $\Gamma_{\rm SC}$. We consider the resistance of the wire network obtained by replacing each edge in $\widetilde{F}_n$ by a wire of resistance $\frac{1}{2}$. (The resistance across a unit cube is then 1.) Let R_n be the effective resistance across $\widetilde{F}_n$, defined as follows. Let $S_i = \{x = (x_1, \ldots, x_d) \in \widetilde{F}_n : x_1 = 3^n i\}$, $i = 0, 1$ be two opposing faces of $\widetilde{F}_n$. Then

$$R_n^{-1} = \inf\{\int_{\widetilde{F}_n} |\nabla f|^2 : f|_{S_i} = i\}. \tag{3.2}$$

There is a dual formulation of resistance in terms of (electric) currents:

$$R_n = \min\{ \text{ energy dissipation due to a current of flux 1 from } S_0 \text{ to } S_1\}. \tag{3.3}$$

(See [**7**] for a more precise definition.)

Let f_n be the optimal (i.e. minimizing) function in (3.2) and I_n be the optimal current in (3.3). Given I_n and I_m we can build a current J_{n+m} which satisfies the conditions of (3.3) for $\widetilde{F}_{n+m}$, with energy less than $c_1 R_n R_m$. Using (3.3) this implies that $R_{n+m} \leq c_1 R_n R_m$. A similar construction with potentials (i.e. functions) proves that $R_{n+m}^{-1} \leq c_2 R_n^{-1} R_m^{-1}$. Thus we obtain:

LEMMA 3.2 (See [**7**] for $d=2$, [**41**] for $d \geq 3$).

$$c_3^{-1} R_n R_m \leq R_{n+m} \leq c_3 R_n R_m. \tag{3.4}$$

So $x_n = \log(c_3 R_n)$ is subadditive, $y_n = \log(R_n/c_3)$ is superadditive, and it follows that there exists $\rho = \rho_d$ (the resistance scaling factor of F) such that

$$c_3^{-1} \rho^n \leq R_n \leq c_3 \rho^n. \tag{3.5}$$

Two open problems here are:

(1) Does $\rho^{-n} R_n \to c_4$ as $n \to \infty$?
(2) Find a way of characterising ρ.

The problem (2) arises because the subadditivity argument just gives the existence of ρ. One can obtain bounds on ρ via bounds on the resistances R_n – see Proposition 5.1 of [**9**]. For example, easy shorting and cutting arguments prove that

$$\frac{2}{3^{d-1}} + \frac{1}{3^{d-1}-1} \leq \rho_d \leq \frac{3}{3^{d-1}-1}. \tag{3.6}$$

(Computer calculations of R_n for $n = 1, \ldots 8$ suggest that $\rho_2 \simeq 1.251$).

We now define, for the d-dimensional SC, the time scaling factor T_d by $T_d = \rho_d M_d$, and

$$\beta_d = \beta_{SC(d)} = \frac{\log \rho_d M_d}{\log L}.$$

LEMMA 3.3. *Let $T_n = \min\{r : X_r \in S_1\}$.*
(a) For $x \in S_0$,

$$\mathbb{E}^x T_n \asymp (M_d \rho_d)^n. \tag{3.7}$$

(b) Γ_{SC} satisfies (E_β).

PROOF. (a) A variant of the argument used to prove Theorem 2.10 shows that there exists a probability measure π on S_0 such that

$$\int_{S_0} \mathbb{E}^x T_n \pi(dx) = R_n M_d^n.$$

Using the elliptic Harnack inequality one can prove that $\mathbb{E}^x T_n$, $x \in S_0$, are all comparable, proving (3.7).
(b) then follows by the usual kind of approximation of balls by cubes. □

THEOREM 3.4. *The bounds $HK(\beta)$ hold for graphical Sierpinski carpets.*

PROOF. This is immediate from Theorems 2.11 and 3.1, and Lemma 3.3. □

REMARK. Theorem 2.4 implies that X is recurrent when $\alpha \leq \beta$, and transient when $\alpha > \beta$. It is clear that $SC(2)$ is recurrent, since it is a subgraph of $\mathbb{Z}^2$. The bounds (3.6) imply that $2.002 \leq \beta_3 \leq 2.073$, while $\alpha_3 = \log 26/\log 3 \simeq 2.966$. Thus $SC(3)$ is transient. (See Section 8 of [**9**] for an example of a recurrent GSC in $\mathbb{R}^3$ with $\alpha > 2$.)

4. General results on pre-fractal graphs

I will now discuss the proof of Theorem 2.11, or rather the implication

$$V_\alpha + E_\beta + EHI \Rightarrow HK(\beta).$$

As usual for heat kernel estimates, one proceeds in the order:

1. On-diagonal upper bounds: $q_t(x,x) \leq ct^{-\alpha/\beta}$,
2. Off-diagonal upper bounds: $q_t(x,y) \leq ct^{-\alpha/\beta}\exp(-c(d(x,y)^\beta/t)^{1/(\beta-1)})$,
3. Lower bounds: $q_t(x,y) \geq ct^{-\alpha/\beta}\exp(-c(d(x,y)^\beta/t)^{1/(\beta-1)})$

In this outline I will concentrate on the points where the anomalous or fractal case $\beta > 2$ differs from the usual one, and will on the whole describe the approach of [**5**]–[**10**] rather than that of [**23**], [**24**]. To avoid minor difficulties I will look at the continuous time random walk Y_t and its transition density $q_t(x,y)$. (See (5.1) for a precise statement of the bounds, and note that they only hold for $t \geq 1 \vee d(x,y)$.)

4.1. On-diagonal upper bounds. The usual argument involves either a Sobolev or Nash inequality, which can be derived from a Poincaré inequality. It would be useful if one could obtain, say, PI(β) directly from EHI and E_β, but I do not know of any general argument. (There is a direct proof of a Poincaré inequality for a class of fractals including some Sierpinski carpets in [**38**], but this argument uses the symmetries of the spaces.)

Instead, one estimates Green's functions for Y. Since $q_t(x,x) \sim t^{-\alpha/\beta}$, $t \geq 1$, one has three cases, which need to be handled separately:

$\alpha < \beta$	(as for SG, SC(2))	Y is (strongly) recurrent,
$\alpha = \beta$		Y is (just) recurrent,
$\alpha < \beta$	(as for SC(d), $d \geq 3$)	Y is transient.

I begin with the first case, $\alpha < \beta$. Let $g_B(x,y)$ be the Green function for $B \subset G$. This satisfies

$$\Delta_y g_B(x,y) = -\delta_x(y),$$

with $g_B(x,z) = 0$ for $z \notin B$. We estimate g_B for $B = B(x_0,R)$.

LEMMA 4.1. *Let $B = B(x_0,R)$. Then $g_B(x_0,x_0) \asymp R^{\beta-\alpha}$.*

PROOF. We have, if $x \in B(x_0, \frac{1}{2}R)$,

$$\sum_{y \in B} g_B(x,y)\mu_0(y) = \mathbb{E}^x \tau_B \asymp R^\beta, \tag{4.1}$$

so as $g_B(x,.)$ attains its maximum at x,

$$cR^\beta \leq \mu_0(B) g_B(x,x) \leq cV(x,R) g_B(x,x) \leq cR^\alpha g_B(x,x).$$

The other bound is a bit more work. Let $B_k = B(x_0, 2^k)$. We can use the EHI to show that $g_B(x_0, y) \asymp g_B(x_0, y')$ if $y, y' \in A = B(x_0, \frac{2}{3}2^k) - B(x_0, 2^{k-2})$. Choose a point $y_1 \in A$. Then (ignoring constants)

$$(2^k)^\beta \geq \sum_{y \in B_k} g_{B_k}(x, y)\mu_0(y) \geq \sum_{y \in A} g_{B_k}(x, y)\mu_0(y) \geq \mu_0(A) g_{B_k}(x_0, y_1),$$

so that, using (V_α),

$$g_{B_k}(x_0, y_1) \leq (2^k)^{\beta - \alpha}. \tag{4.2}$$

Let $S = \min\{t : Y_t \notin B_{k-1}\}$. Then

$$g_{B_k}(x_0, x_0) = g_{B_{k-1}}(x_0, x_0) + \mathbb{E}^{x_0} g_{B_k}(Y_S, x_0) \leq g_{B_{k-1}}(x_0, x_0) + (2^k)^{\beta-\alpha}.$$

Iterating it follows that $g_{B_k}(x_0, x_0) \leq c(2^k)^{\beta-\alpha}$. □

Let $g_\lambda(x, y)$ be the λ-potential density, so that

$$g_\lambda(x, y) = \int_0^\infty e^{-\lambda s} q_s(x, y) ds = \int_0^T q_s(x, y) ds,$$

where T is an independent negative exponential r.v. with mean λ^{-1}. Using (E_β) we have $\mathbb{E}^x \tau(x, R) \asymp R^\beta$, so one would expect (and, using EHI, can prove) that if $\lambda^{-1} = R^\beta$ then

$$g_\lambda(x, x) \asymp g_{B(x,R)}(x, x) \asymp R^{\beta-\alpha} = \lambda^{\alpha/\beta - 1}.$$

We then have, taking $\lambda = t^{-1}$, and using the fact that $q_\cdot(x, x)$ is decreasing,

$$t^{1-\alpha/\beta} \asymp g_\lambda(x, x) \geq \int_0^t e^{-s/t} q_s(x, x) ds \geq (1 - e^{-1}) t q_t(x, x),$$

which gives the global upper bound

$$q_t(x, y) \leq q_t(x, x)^{1/2} q_t(y, y)^{1/2} \leq c t^{-\alpha/\beta}, \quad t \geq 1. \tag{4.3}$$

The case $\alpha \geq \beta$ again uses estimates on the Greens functions. However, since the process is transient, one has $g(x, x) = O(1)$ so that g_B or g_λ gives no extra information. One can instead look at

$$g_{\lambda,p}(x, x) = \int_0^\infty t^p e^{-\lambda s} q_s(x, x) ds,$$

which one estimates in a similar way. For details see [**9**].

4.2. Off-diagonal upper bounds. The classical method to obtain these is 'Davies method' of perturbed semigroups, but this has not yet been made to work in the anomalous diffusion context. Instead we use a probabilistic type of chaining argument. For more details see [**8**] or Chapter 3 of [**3**].

LEMMA 4.2 (([**5**], Lemma 1.1)). *Let $\xi_1, \xi_2, \ldots, \xi_n$, V be non-negative r.v. such that $V \geq \sum_1^n \xi_i$. Suppose that for some $p \in (0, 1)$, $a > 0$,*

$$\mathbb{P}\big(\xi_i \leq t | \sigma(\xi_1, \ldots, \xi_{i-1})\big) \leq p + at, \qquad t > 0. \tag{4.4}$$

Then

$$\log \mathbb{P}(V \leq t) \leq 2\left(\frac{ant}{p}\right)^{1/2} - n \log \frac{1}{p}. \tag{4.5}$$

PROOF. If $\eta \ge 0$ is a r.v. with distribution function $P(\eta \le t) = (p + at) \wedge 1$, then

$$E\left(e^{-\lambda \xi_i} | \sigma(\xi_1, \ldots, \xi_{i-1})\right) \le E e^{-\lambda \eta} = p + \int_0^{(1-p)/a} e^{-\lambda t} a dt \le p + a\lambda^{-1}.$$

So

$$\begin{aligned} \mathbb{P}(V \le t) &= \mathbb{P}\left(e^{-\lambda V} \ge e^{-\lambda t}\right) \le e^{\lambda t} \mathbb{E} e^{-\lambda V} \\ &\le e^{\lambda t} \mathbb{E}(\exp \lambda \sum_1^n \xi_i) \le e^{\lambda t}(p + a\lambda^{-1})^n \le p^n \exp\left(\lambda t + \frac{an}{\lambda p}\right). \end{aligned}$$

The result follows on setting $\lambda = (an/pt)^{1/2}$. □

Now let $R \ge 1$ and $t \le cR^\beta$. We use Lemma 4.2 to estimate $\mathbb{P}^x(\tau(x, R) \le t)$. Choose $n \ge 1$, $r = R/n$, and let the ξ_i be successive times taken by Y to move a distance r. So if $\sum_1^k \xi_i = T_k$ then

$$\xi_{k+1} = \tau(Y_{T_k}, r) - T_k, \quad k \ge 0.$$

Clearly $\sum_1^{cn} \xi_i \le \tau(x, R)$. Using (E_β) we have that there exists $p_0 < 1$ such that

$$\mathbb{P}(\xi_i < s) \le p_0 + \frac{c_1 s}{r^\beta}.$$

So

$$\log \mathbb{P}^x(\tau(x, R) \le t) \le 2\Big(\frac{c_1 n^\beta}{R^\beta} \frac{nt}{p_0}\Big)^{1/2} - n \log \frac{1}{p_0} = cn\Big(\big(\frac{n^{\beta-1} t}{R^\beta}\big)^{1/2} - c_2\Big).$$

Up to a constant, this is minimised by taking $n^{\beta-1} = cR^\beta/t$. We deduce, as long as $r \ge 1$, so that $t \ge cR$,

$$\mathbb{P}^x(\tau(x, R) \le t) \le e^{-cn} \le \exp(-c(R^\beta/t)^{1/(\beta-1)}). \tag{4.6}$$

We now wish to combine the bound (4.6) with the global upper bound (4.3). Using the symmetry of Y and the Markov property this can be done quite easily. Fix x and y, let $R = \frac{1}{2} d(x, y)$ and

$$A_x = \{z : d(z, x) \le d(z, y)\}, \quad A_y = G - A_x.$$

Then

$$\mu_0(y) q_t(x, y) = \mathbb{P}^x(Y_t = y) = \mathbb{P}^x(Y_t = y, Y_{t/2} \in A_x) + \mathbb{P}^x(Y_t = y, Y_{t/2} \in A_y). \tag{4.7}$$

To bound the second term in (4.7) we use the Markov property:

$$\begin{aligned} \mathbb{P}^x(Y_t = y, Y_{t/2} \in A_y) &= \mathbb{P}^x(\tau(x, R) \le t/2, Y_{t/2} \in A_y, Y_t = y) \\ &\le \mathbb{P}^x(\tau(x, R) \le t/2) \sup_z \mathbb{P}^z(Y_{t/2} = y) \\ &\le c \exp(-c(R^\beta/t)^{1/(\beta-1)})(t/2)^{-\alpha/\beta}; \end{aligned}$$

we used (4.6) and (4.3) in the final line.

By symmetry

$$\mu_0(x) \mathbb{P}^x(Y_t = y, Y_{t/2} \in A_x) = \mu_0(y) \mathbb{P}^y(Y_t = x, Y_{t/2} \in A_x),$$

so the first term in (4.7) can be bounded in exactly the same way.

REMARK. It may seem surprising that bounds obtained in this way are (up to constants) of the right order. After all, in controlling the escape time term $\mathbb{P}^x(\tau(x,R) \le t)$ we made no attempt to examine which direction the process moved in each of the n short time steps. However, the improvement to the global upper bound $t^{-\alpha/\beta}$ is only relevant for t such that $t \ll R^\beta$. In this case the process Y is very unlikely to reach y, and if it does it has to move more or less directly on the path from x to y.

It is easy to obtain an 'on-diagonal' lower bound from (4.6). Choose c_1 so that $\mathbb{P}^x(Y_t \notin B(x, c_1 t^{1/\beta})) < \frac{1}{2}$. Then

$$\tfrac{1}{4} < \left(\int_{B(x,c_1 t^{1/\beta})} q_t(x,y) d\mu_0 \right)^2 \le \int_B q_t(x,y)^2 d\mu_0 \int_B d\mu_0 \le q_{2t}(x,x)\mu_0(B),$$

from which it follows that

$$q_t(x,x) \ge ct^{-\alpha/\beta}, \quad t \ge 1. \tag{4.8}$$

4.3. Lower bounds. The key step here is extending the lower bound (4.8) to a 'near diagonal' lower bound:

$$q_t(x,y) \ge c_1 t^{-\alpha/\beta}, \quad y \in B(x, c_2 t^{1/\beta}),\ t \ge 1. \tag{4.9}$$

A standard chaining argument using the Markov property can then be used to prove the full lower bound. Fix x, y and t, let $R = d(x,y)$, neglect constants, and as in the previous chaining argument set $n^{\beta-1} = R^\beta/t$. Let $x = z_0, z_1, \dots, z_n = y$ be a chain of points on the geodesic between x and y with $d(z_{i-1}, z_i) = R/n$. If $s = t/n$ and $r = R/n$ then the choice of n implies that

$$\frac{r^\beta}{s} = \frac{R^\beta}{n^\beta}\frac{n}{t} = 1.$$

Let $B_i = B(z_i, r)$; using (4.9) and (V_α)

$$q_s(x', y') \ge c_1 \mu_0(B_i)^{-1}, \quad x' \in B_{i-1},\ y' \in B_i.$$

So

$$\begin{aligned} q_t(x,y) &\ge \int_{B_1} \cdots \int_{B_{n-1}} q_s(x, z_1) \dots q_s(z_{n-1}, y)\mu_0(dz_1) \dots \mu_0(dz_{n-1}) \\ &\ge c_1^{n-1} \mu_0(B_n)^{-1} \ge s^{-\alpha/\beta} \exp(-c(R^\beta/t)^{1/(\beta-1)}), \end{aligned}$$

giving the lower bound in $HK(\beta)$.

To extend (4.8) to (4.9) one uses the elliptic Harnack inequality. (In [**9**], working with Sierpinski carpets, we were able to use a coupling argument, but this does not generalise.) See [**8**] or [**24**] for details.

4.4. Stability results for general pre-fractal graphs. A property $\mathcal{P}$ of graph is *stable under rough isometries* (or just 'stable'), if, given two roughly isometric graphs Γ_1 and Γ_2, then $\mathcal{P}$ holds for Γ_1 if and only if it holds for Γ_2. It is clear that (V_α) (and also the weaker condition (VD)) are stable. Inequalities involving the Dirichlet form $\mathcal{E}(f,f)$ are in general stable. Denote by WPI(β) the weak Poincaré inequality

$$\int_{B(x,R)} (f - \overline{f})^2 \le cR^\beta \int_{B(x,\lambda R)} |\nabla f|^2,$$

for some $\lambda \geq 1$. Then WPI(β) is stable. Since (see [Je]) it is known that (VD) plus WPI(β) implies PI(β), it follows that (VD) plus PI(β) is stable. It is an open problem whether the conditions (E_β) and (EHI) are stable.

One consequence of Theorem 2.8 is that the standard parabolic Harnack inequality PHI(2), and the bounds HK(2) are stable – this is clear since they are equivalent to the two stable conditions (VD) and PI(2). It is natural to ask if the bounds HK(β) are similarly stable, and if so what (stable) conditions are equivalent to them. A first guess might be that a simple extension of Theorem 2.8 to the $\beta > 2$ case holds – so that HK(β) is equivalant to (VD) and PI(β). However, this is easily seen to be false.

First note that PI(β) gets weaker as β increases, so that if $\beta_1 < \beta_2$ then PI(β_1) implies PI(β_2). Consider the product graph $\Gamma = \mathbb{Z} \times G$, where G is the graphical Sierpinski gasket. Then PI(β_{SG}) holds for both $\mathbb{Z}$ and Γ_{SG}, and one can verify that it also holds for the product. But one can write the continuous time heat kernel q_t for Γ as a product:

$$q_t((n_1, x_1), (n_2, x_2)) = q_{\mathbb{Z},t}(n_1, n_2) q_{SG,t}(x_1, x_2), \quad n_i \in \mathbb{Z},\, x_i \in G,$$

and it follows that HK(β_{SG}) must fail for Γ. The difficulty is that PI(β) only controls the slowest rate of homogenisation of q_t in a ball B.

In a recent paper [**12**] Bass and I have proved stabilty of HK(β).

Definition. Let $\beta \geq 2$. We say Γ satisfies CS(β), a *cut-off Sobolev inequality with exponent* β, if there exists $\theta \in (0, 1]$, and constants c_1 and c_2 such that for every $x_0 \in G$, $R \geq 1$, there exists a cut-off function $\varphi (= \varphi_{x_0,R})$ satisfying properties (a)–(d) below.
(a) $\varphi(x) \geq 1$ for $x \in B(x_0, R/2)$.
(b) $\varphi(x) = 0$ for $x \in B(x_0, R)^c$.
(c) $|\varphi(x) - \varphi(y)| \leq c_1 (d(x, y)/R)^\theta$ for all x,y.
(d) For any ball $B(x_1, s)$ with $1 \leq s \leq R$ and $f : B(x_1, 2s) \to \mathbb{R}$,

$$\begin{aligned} &\sum_{x \in B(x_1,s)} f(x)^2 \sum_{y \in G} a_{xy} |\varphi(x) - \varphi(y)|^2 \\ &\leq c_2 (\frac{s}{R})^{2\theta} \Big(\sum_{x,y \in B(x_1,2s)} a_{xy} |f(x) - f(y)|^2 + s^{-\beta} \sum_{x \in B(x_1,2s)} \mu_0(x) f(x)^2 \Big). \end{aligned} \tag{4.10}$$

We call φ a *cut-off function*, and (4.10) a weighted Sobolev inequality relative to φ, with exponent β and scale R.

Theorem 4.3 ([**12**]). *Suppose Γ satisfies the (p_0) condition. The following are equivalent:*
(a) Γ satisfies (VD)+(EHI)+(E_β).
(b) Γ satisfies PHI(β).
(c) Γ satisfies HK(β).
(d) Γ satisfies (VD), PI(β) and $CS(\beta)$.

The equivalence between (a), (b) and (c) has already been given in Theorem 2.11. The proof that (c) implies (d) is done by using properties of Green's functions to construct a suitable cut-off function φ. The hardest part of the proof that (d) implies (a) is that of the EHI; this is done by an iterative argument as in Moser [**44**]. The condition CS(β) looks quite complicated, but it can be proved to be stable under rough isometries – see [**27**]. The essential point of this inequality is the following.

Many heat kernel arguments (such as Moser's proof of the elliptic Harnack inequality in [**44**]) use at some point a 'cut-off' function φ. In most classical contexts a linear function is adequate, and one uses the easy estimate

$$I(R,f) = \int_{B(x_0,R)} f^2 |\nabla\varphi|^2 \le ||\nabla\varphi||_\infty^2 \int_{B(x_0,R)} f^2. \tag{4.11}$$

If $f = 1$ then this bounds $I(R,f)$ by $cR^{-2}\mu_0(B(x_0,R))$, which is the right order for $R_{\text{eff}}(B(x_0,R/2), B(x_0,R)^c)^{-1}$.

However, in the fractal context one expects that

$$R_{\text{eff}}(B(x_0,R/2), B(x_0,R)^c) \asymp \frac{R^\beta}{\mu_0(B(x_0,R))};$$

the 'holes' lead to a higher resistance or lower conductance. Working through Moser's argument one finds that one needs to control $I(R,f)$ by a term of the order of $(\inf_\varphi \int |\nabla\varphi|^2)(\int f^2)$, so that (4.11) is too weak. The condition CS(β) gives the existence of enough 'low energy' cut-off functions to make the Moser argument work.

Remarks. 1. Using (4.11) one sees that CS(2) always holds – it has in effect been used many times in the literature, but since it was trivial no one noticed.
2. If $\beta_1 < \beta_2$ then it is clear from (4.10) that CS(β_2) implies CS(β_1).
3. It is quite possible that a (simpler) condition than CS(β) might be found which would (with VD and PI(β)) be equivalent to HK(β).
4. Using Theorem 4.3, and the stability of (VD) plus PI(β) and CS(β) under rough isometries, one can extend Theorem 2.4 to more general pre-fractal graphs. It should also be possible to study Brownian motion on some classes of pre-fractal manifolds using the same ideas.

To make more explicit the connection between PI(β), CS(β) and the effective resistance of annuli, I give here an (easy) result from [**12**].

Lemma 4.4. *Let Γ satisfy (p_0) and (VD).*
(a) If PI(β) holds, then

$$R_{\text{eff}}(B(x_0,R), B(x_0,2R)^c) \le c_1 \frac{R^\beta}{\mu_0(B(x_0,R))}.$$

(b) If CS(β) holds, then

$$R_{\text{eff}}(B(x_0,R), B(x_0,2R)^c) \ge c_2 \frac{R^\beta}{\mu_0(B(x_0,2R))}.$$

(c) If G_C satisfies CS(β_1) and PI(β_2), then $\beta_1 \le \beta_2$.

Proof. Write $V_0 = \mu_0(B(x_0,R))$. Using (VD) one can prove that $V_0 \asymp \mu_0(B(x,cR))$ for any $x \in B(x_0,3R)$. For (a) apply PI(β) in the ball $B(x_0,3R)$ to the function f which minimises $R_{\text{eff}}(B(x_0,R), B(x_0,2R)^c)$. Then

$$V_0 \asymp \int |f-\overline{f}|^2 \le cR^\beta \int |\nabla f|^2 = cR^\beta R_{\text{eff}}^{-1}.$$

(b) Let φ be a cut-off function for $B(x_0,R)$ given by CS(β). Then taking $f=1$ and using $B(x_0,R)$ in (4.10) we obtain

$$R_{\text{eff}}(B(x_0,R/2),B(x_0,R)^c)^{-1} \le \int_{B(x_0,R)} |\nabla\varphi|^2 d\mu_0 \le cR^{-\beta}\int_{B(x_0,2R)} f^2 \le c\frac{V_0}{R^\beta}.$$

(c) is immediate from (a) and (b). □

5. Continuum limits

I will now discuss briefly the continuum limits of the random walks of section 2. I will continue to work in more or less the simplest framework: that of a GSG or GSC. It is slightly simpler to give the construction in the case of the unbounded fractal $\widetilde{F}$

Let $\Gamma=(G,E)$ be a pre-fractal graph associated with $\{\psi_1,\dots,\psi_M\}$. If we set $\Phi(r,t)=(r^\beta/t)^{1/(\beta-1)}$ then we can write the bounds on q_t as:

$$\begin{aligned}c_1t^{-\alpha/\beta}\exp\Big(-c_2\Phi(d(x,y),t)\Big) &\le q_t(x,y)\\ &\le c_3t^{-\alpha/\beta}\exp\Big(-c_4\Phi(d(x,y),t)\Big),\quad t\ge 1\vee d(x,y).\end{aligned} \tag{5.1}$$

(If $t<d(x,y)$ then $q_t(x,y)$ is not zero, but the bounds take a different form.) I will assume that Γ has the property that the graph metric $d(x,y)$ and the Euclidean metric are comparable. (If not, the same general principles will apply, but we need to take a bit of care with the metric on F.) So we can write $|x-y|$ instead of $d(x,y)$ in (5.1).

Let $G_n=\psi_1^n(G)=L^{-n}G\subset\mathbb{R}^d$, let

$$Y_t^{(n)}=L^{-n}Y_{L^{n\beta}t},\quad t\ge 0, \tag{5.2}$$

and $\mathbb{P}_n^x$ be the probability law of the process $Y^{(n)}$ started at x. Let $\mu_n(A)=L^{-n\alpha}\mu_0(L^nA)$ for $A\subset G_n$. If we write $q_t^{(n)}(x,y)$ for the transition density of $Y^{(n)}$ then

$$\begin{aligned}q_t^{(n)}(x,y)&=\mu_n(y)^{-1}\mathbb{P}_n^x(Y_t^{(n)}=y)=L^{n\alpha}\mu_0(L^ny)^{-1}\mathbb{P}^{L^nx}(Y_{L^{n\beta}t}=L^ny)\\ &=L^{n\alpha}q_{L^{n\beta}t}(L^nx,L^ny)\\ &\le c_1\frac{L^{n\alpha}}{(L^{n\beta}t)^{\alpha/\beta}}\exp(-c_2\Phi(|L^nx-L^ny|,L^{n\beta}t))\\ &=c_1t^{-\alpha/\beta}\exp(-c_2\Phi(|x-y|,t)),\end{aligned} \tag{5.3}$$

with a similar lower bound. Note that this bound holds when

$$L^{n\beta}t\ge 1,\ \text{and}\ L^n|x-y|\le L^{n\beta}t, \tag{5.4}$$

so that as $n\to\infty$ we obtain bounds for all x, y, and $t>0$.

Let the resolvent of $Y^{(n)}$ be given by

$$U_\lambda^{(n)}f(x)=\mathbb{E}^x\int_0^\infty e^{-\lambda t}f(Y_t^{(n)})dt=\int_{G_n}\int_0^\infty e^{-\lambda t}q_t^{(n)}(x,y)dtd\mu_n(y).$$

We can check (using (5.3)) that $U_\lambda^{(n)}$ maps $L^2(G_n,\mu_n)$ into itself. As G_n is a discrete space $U_\lambda^{(n)}$ is also a map on $C_0(G_n)$.

PROPOSITION 5.1. *(a)* $||U_\lambda^{(n)} f||_\infty \le \lambda^{-1}||f||_\infty$.
(b) For $f \in C_0(G_n)$ $\lambda U_\lambda^{(n)} f \to f$ *in* $C_0(G_n)$ *as* $\lambda \to \infty$.
(c) $U_\lambda^{(n)}$ *satisfies the resolvent equation:*

$$U_\lambda^{(n)} - U_{\lambda'}^{(n)} = (\lambda - \lambda')U_\lambda^{(n)} U_{\lambda'}^{(n)}. \tag{5.5}$$

(d) There exists $\theta \in (0,1]$ *and* $c(\lambda)$ *independent of* n *such that for* $x, y \in G_n$ *with* $|x - y| \le 1$,

$$|U_\lambda^{(n)} f(x) - U_\lambda^{(n)} f(y)| \le c(\lambda)|x-y|^\theta ||f||_\infty. \tag{5.6}$$

PROOF. Of these (a)–(c) are standard properties of resolvents. The proof of (d) uses the elliptic Harnack inequality. The essential idea is to take $r \ll \lambda^{-1}$, and work in $B = B(x_0, r)$. Writing τ for the first exit time of $Y^{(n)}$ from B, we have for $x \in B$, and some g,

$$U_\lambda^{(n)} f(x) = \mathbb{E}^x U_\lambda^{(n)} f(Y_\tau^{(n)}) + g(x).$$

The first function is harmonic in B, while the second can be bounded by $cr^\beta ||f||_\infty$. Since the EHI implies that harmonic functions are Hölder continuous of some order θ depending only on the constant c_1 in the EHI, this leads to (d). □

The natural approach to the construction of a diffusion Z on the limit fractal $F = F_\Psi$ is to take a limit of the approximating random walks $Y^{(n)}$. The estimates (5.3) are enough to prove weak compactness of the probability laws $\mathbb{P}_n^x$. However, to prove that the limits can be combined to give a Markov process, it is more convenient to use compactness of resolvents.

THEOREM 5.2. *There exists a subsequence* (n_k) *such that* $V^\lambda f = \lim_k U_\lambda^{(n_k)} f$ *exists for each* $\lambda > 0$ *and continuous* f *on* $\widetilde{F}$. *The family* (V^λ) *is the resolvent of a continous Feller diffusion* $Z = (Z_t, t \ge 0, \mathbb{P}^x, x \in \widetilde{F})$. Z *has a* μ*-symmetric semigroup* T_t *on* $C_0(\widetilde{F})$, *and a symmetric transition density* $r_t(x,y)$, $x, y \in \widetilde{F}$ *which satisfies*

$$c_1 t^{-\alpha/\beta} e^{-c_2 \Phi(|x-y|,t)} \le r_t(x,y) \le c_3 t^{-\alpha/\beta} e^{-c_4 \Phi(|x-y|,t)},\ t > 0,\ x, y \in \widetilde{F}. \tag{5.7}$$

PROOF. The existence of V^λ is proved by a diagonalization argument using the estimates in Proposition 5.1. This also gives that (V^λ) satisfies the resolvent equation (5.5), and the estimate (5.6). Let Z be the Markov process associated with (V^λ). The uniqueness of the Laplace transform implies that $Y^{(n_k)}$ converge weakly to Z, and (5.7) then follows from (5.3) and (5.4). The symmetry of Z is an easy consequence of the symmetry of $Y^{(n)}$. □

REMARKS. 1. It is natural to ask if one has

$$Y^{(n)} \Rightarrow Z \text{ as } n \to \infty, \tag{5.8}$$

rather than just convergence along a subsequence. This does hold for finitely ramified fractals (p.c.f. self-similar sets), and is proved using the decimation properties of these spaces. It then follows that

$$(L_F Z_t, t \ge 0) \overset{(d)}{=} (Z_{T_F t}, t \ge 0), \tag{5.9}$$

where T_F is the time scale factor of the fractal F.
2. For SCs it is not known if (5.8) holds. By working a bit harder (more diagonalization etc.) one can however build a limit Z with the scaling property (5.9) – see [**38**].
3. The limiting process Z inherits all the symmetries of the random walk X on Γ. For the SG one can prove (see [**14**]) that any diffusion W on F_{SG} satisfying certain symmetries (which have to be a bit more than one might initially guess) is, up to a constant time change, the same as Z.
4. For the SC there are actually two different levels of uniqueness one might ask for, both open.

(1) Prove that the laws $(\mathbb{P}_n^x)$ of $(Y^{(n)})$ have just one cluster point.
(2) Characterise the limit Z as the unique (up to a constant time change) diffusion on F satisfying certain symmetry conditions.

5. Associated with Z and the semigroup (T_t) is a 'Laplacian' $\mathcal{L}$ and a Dirichlet form $\mathcal{E}$. The domain $\mathcal{D}(\mathcal{E})$ is a Besov type space – see [**31, 35, 25**]. Several unusual properties of $\mathcal{L}$, $\mathcal{E}$ appear to be linked with the anomalous diffusion (that is, the fact that $\beta > 2$), but the exact links are not clear.

(1) It appears that the energy measure ν_f, given formally by $d\nu_f = |\nabla f|^2 d\mu$, is singular with respect to μ. This was proved for the Sierpinski gasket in [**37**], but is conjectured to hold quite generally.
(2) The wave equation on F is expected to have infinite velocity.
(3) The domain of the operator $\mathcal{L}$ is not an algebra – see [**37, 25**].

Open Problems.

I have mentioned a number of open problems in the text. For convenience I list them again here.
1. Find a direct proof (not using heat kernel estimates) that EHI and the resistance bound for annuli

$$R_{\text{eff}}(B(x_0,R),B(x_0,2R)^c) \asymp \frac{R^\beta}{\mu_0(B(x_0,2R))}$$

imply PI(β). (One may have to use (VD)).
2. Prove that $\rho^{-n}R_n$ converge, where R_n are the resistances across cubes of side L^n in Γ_{SC}.
3. Prove the uniqueness of the diffusion on the SC.
4. Find a simpler condition than CS(β) which can be used in Theorem 4.3(d).
5. Prove that μ and ν_f are mutually singular for Sierpinski carpets.

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