Let $\mathcal{A}$ be an algebra. Suppose that whenever $(A_n, n \in \mathbb{N})$ are disjoint sets in $\mathcal{A}$ the set $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. Prove that $\mathcal{A}$ is a $\sigma$-algebra.

If $A \in \mathcal{A}$ then $A^c \in \mathcal{A}$.

Let $(A_n)$ be set in $\mathcal{A}$. Set $B_1 = A_1$, $B_n = A_n \setminus \bigcup_{j=1}^{n-1} A_j$.

Then $\bigcup_{j=1}^{n} B_n = \bigcup_{j=1}^{n} A_n$, and so $\bigcup_{j=1}^{n} B_j = \bigcup_{j=1}^{n} A_j$.

The sets $B_n$ are disjoint, since if $j < n$ then $B_j \subseteq \bigcup_{i=1}^{j-1} A_i = \bigcup_{i=1}^{n-1} A_i$.

and $B_n \cap \bigcap_{i=1}^{n-1} A_i = \emptyset$ by the construction of all $B_n$.

Since $\mathcal{A}$ is an algebra, $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. By the property of $\mathcal{A}$, $\bigcup_{n=1}^{\infty} B_n \in \mathcal{A}$, and so $\bigcup_{n=1}^{\infty} A_n \in \mathcal{A}$. 

Sample Solution
2. (a) Define \sigma-\text{finite measure, semi-finite measure.}

A measure \( \mu \) on a space \( (X, \mathcal{M}) \) is \( \sigma-\)finite if

\[
\exists \{A_n\} \text{ such that } A_n \in \mathcal{M}, \quad \mu(A_n) < \infty \quad \forall n \quad \text{and} \quad \bigcup_{n=1}^{\infty} A_n = X.
\]

\( \mu \) is semi-finite if whenever \( F \in \mathcal{M} \) and \( \mu(F) = \infty \) there exists \( E \subseteq F \) with \( \mu(E) < \infty \).

(b) Let \( X \) be a countable set and \( \mu \) be a semi-finite measure on \( \mathcal{P}(X) \). Is \( \mu \) \( \sigma \)-finite? Prove your answer or give a counterexample.

Let \( X = \{x_n, n \in \mathbb{N}\} \). Let \( \mu \) be semi-finite.

Let \( n \geq 1 \). Suppose \( \mu(\{x_n\}) = \infty \). The only subsets of \{x_n\} are \( \emptyset \) and \( \{x_n\} \), and \( \mu(\emptyset) = 0 \) and \( \mu(\{x_n\}) = \infty \).

So \( \mu \) is not semi-finite, a contradiction.

Thus we have \( \mu(\{x_n\}) < \infty \) for each \( n \).

Then \( X = \bigcup_{n=1}^{\infty} \{x_n\} \), and so \( \mu \) is \( \sigma-\)finite.