Research Statement

Introduction  My research is in mathematical physics, specifically in questions relating quantum field theory, representation theory and geometry. My thesis involves understanding the analytic aspects of $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories. I give a mathematically rigorous proof of the conjecture (made in [5]) that for any compact gauge group, the partition function of an $\mathcal{N} = 2$ Super-Chern-Simons matter theory on $S^3$ factorizes into a finite sum over new structures called “holomorphic blocks.” These blocks are conjectured to be contact structure invariants, giving a three dimensional analog of conformal blocks. The factorization is thought to be related to the Heegaard splitting of $S^3$ into a pair of solid tori.

To prove this factorization, I use integrals of hyperbolic hypergeometric functions called double sine functions (or quantum dilogarithms) to study certain super symmetric partition functions on $S^3$. In recent years, the double sine function has seen interest for example in [20], [10] and [19] and their integrals have been explored extensively in [7]. In my thesis work, I show that integrals of double sine functions are intimately related to the global parity anomaly in three dimensional gauge theory. Along the way, I extend the higher dimensional Jordan lemma of [23] to meromorphic forms in dimension $d > 2$ with more than $d$ generic divisors.

My future work is focused on understanding the relation between integrals of double sine functions, three-dimensional holomorphic blocks, and the geometry of three manifolds.

Summary of current work. Much of the following work is contained in a 230 page joint paper with C. Beasley in preparation [4]. This paper can be found on my personal website.

Let $G$ be a compact Lie group, $\mathfrak{h}$ its Cartan subalgebra and $\Lambda = R_1 \oplus \ldots \oplus R_n$ a decomposable representation of $G$. Denote by $c_2(\Lambda)$ the trace of the quadratic Casimir operator acting on $\Lambda$. We normalize the Casimir by the condition that the trace on the adjoint is twice the dual Coxeter number $c_2(\mathfrak{g}) = 2h_\mathfrak{g}$.

The most basic object of study in a quantum field theory is called the partition function $Z$. The partition function is often defined as a formal infinite dimensional integral over the space of all field configuration. When the quantum theory is super symmetric, it is possible that $Z$ can be “localized” to a sum or finite dimensional integral by an analog of the method of stationary phase. This expresses $Z$ as a well defined mathematical object that can be rigorously studied.

In the last five years, it has been shown that the partition functions $Z_{S^3}$ of $\mathcal{N} = 2$ supersymmetric Chern-Simons matter theories with gauge group $G$ on $S^3$ can be written as finite dimensional integrals, see [18], [13]. On a family of squashed three sphere geometries $S^3_b$ with metric given by the embedding

$$b^2|z_1|^2 + b^{-2}|z_2|^2 = 1, \quad b \in \mathbb{R}, \quad (z_1,z_2) \in \mathbb{C}^2,$$

and chiral matter in the representation $\Lambda$ of $G$, the partition function can be written as

$$Z_{S^3}(k, \Lambda) = \int_\mathfrak{h} d^\mathfrak{h} \sigma \exp \left[ - \frac{ik}{4\pi} \text{Tr}(\sigma^2) \right] \times$$

$$\times \prod_{\alpha \in \Delta_+} \left[ 4 \sinh \left( \frac{b(\alpha,\sigma)}{2} \right) \sinh \left( \frac{\langle \alpha,\sigma \rangle}{2b} \right) \right] \prod_{\beta \in \Delta_j} \left[ \sum_{j=1}^n \prod_{\beta \in \Delta_j} s_b \left( \frac{\langle \beta,\sigma \rangle}{2 \pi} + \mu_j \right) \right]. \quad (1)$$

Here, the $\Delta_+$ denote the positive roots of $\mathfrak{g}$, $\Delta_j$ the weights of the representation $R_j$ and the mass parameters $\mu_j$ are complex constants. The functions $s_b(z)$ can be presented as

$$s_b(z) \sim \sum_{m,n=0}^{\infty} \frac{(m+1)b + (n+1)b^{-1} + iz}{mb + nb^{-1} - iz}, \quad b \in \mathbb{R}, \quad z \in \mathbb{C}, \quad (2)$$
Figure 1: Here, we show an example of the representation theoretic data needed to construct $Z_{S^3}(k, \Lambda)$ for the gauge group $G_2$, with matter in the representation $V_{1,0} = 7$. The positive simple roots are $\hat{\alpha}_{1,2}$ and the fundamental weights are $\hat{\omega}_{1,2}$. The weights $\beta$ of the fundamental representation $V_{1,0}$ are circled.

and are known variously as the double sine [20], hyperbolic gamma [7] or quantum dilogarithm [10] functions.

Since the localized partition function (1) was first derived, hundreds of papers have appeared on the arXiv trying to extract theoretical insight from these integrals. However, it was not known how to evaluate $Z_{S^3}(k, \Lambda)$ for representations $\Lambda$ outside of the fundamental and adjoint representation of $G = U(N)$. In my thesis, I show how to evaluate (1) for any gauge group and give a general formula for all representations $\Lambda = R_1 \oplus \ldots \oplus R_n$, where $R_j$ are multiplicity free. Along the way, I derive several interesting results:

First, I show that the shift in Chern-Simons level $k$ due to the spectral flow of the gauge twisted Dirac and Laplace operators (see, for example [25]) can be seen explicitly in the asymptotic behavior of the double sine function. I show that the effect on the partition function $Z_{S^3}(k, R_1 \oplus \ldots \oplus R_j)$ of integrating out a large mass $|\mu_j| \gg 1$ matter multiplet in the representation $R_j$ leads to a shift in Chern-Simons level by

$$\lim_{|\mu_j| \to \infty} e^{\phi(\mu_j)} Z_{S^3}(k, R_1 \oplus \ldots \oplus R_j) = Z_{S^3} \left( k - \text{sign}(\mu_j) \frac{1}{2} c_2(\Lambda), R_1 \oplus \ldots \oplus R_j-1 \right)$$  (3)

Here, $\phi(\mu_j)$ is a quadratic renormalization factor of $\mu_j$ alone and $c_2(R_j)$ is the quadratic Casimir. Physically, this shift must usually be derived from a careful analysis of the spectral flow of the Dirac operator or from sensitive one loop computations. Mathematically, this shift allows us compute integrals (1) in regions of $k$ where the Gaussian Chern-Simons term usually prevents the use of the Jordan lemma.

Second, I show that when the parity anomaly condition $k - \frac{1}{2} c_2(\Lambda) \in \mathbb{Z}$ is satisfied, the partition function on $S^3$ factorizes into a finite sum of hypergeometric functions $B_a(q, x, k)$ called “holomorphic blocks.” These blocks depend on $b$ or $b^{-1}$ through the variables

$$q = e^{2\pi i b^2}, \quad \bar{q} = e^{2\pi i b^{-2}}, \quad x_j = e^{2\pi i b \mu_j}, \quad \bar{x}_j = e^{2\pi i b^{-1} \mu_j},$$  (4)

and $x = (x_1, \ldots, x_n)$. They allow us to write the partition function as

$$Z_{S^3}(k) = \sum_{a \in \mathcal{I}} M_a(k) B_a(q, x, k) B_a(\bar{q}, \bar{x}, k).$$  (5)

Here, index $\mathcal{I}$ parametrizes linearly independent sets of weights in $\Delta_\Lambda$. The holomorphic blocks $B_a(q, k)$ are thought to be an analog of conformal blocks in two dimensions. Such a factorization...
was shown for rank 1 gauge groups in [21] and for \( G = U(N) \) with matter in the fundamental representation in [24]. In both cases, the integrand splits into a product of one dimensional integrals. In [4], I show factorization for any compact Lie group \( G \), and matter in any direct sum of multiplicity free representations.

The factorization in [15] is conjectured in [3, 9] to correspond to a topological fusion along the Heegaard decomposition of \( S^3 \) into two solid tori with \( q \) twisted metric \( D^2 \times_q S^1 \):

\[
D^2 \times_q S^1 \cup_{\tau} D^2 \times_{q} S^1 = S^3_{b} \quad \mathcal{H} \otimes \mathcal{H}^* = Z_{S^3}.
\]

In this setup, \( M_a(k) \) are the elements of a diagonal matrix giving a metric on \( \mathcal{H} \).

Finally, the key step in evaluating (1) for general representations is to prove an extension of the higher dimensional Jordan lemma of [23]. Known techniques did not allow the evaluation of integrals of meromorphic forms \( \omega \) in \( r > 2 \) dimensions with more than \( r \) generic linear divisors. In [4], I show how to extend the Jordan lemma to allow the computation of arbitrarily large numbers of (generic) divisors, given certain conditions on the structure of \( \omega \). In addition to integrals of double sine functions, this lemma has application to Mellin-Barnes type integrals [22], higher dimensional Fourier transforms, and the computation of scattering amplitudes in four dimensional gauge theories [11].

**Future Projects**

I am interested in understanding the structure of the three dimensional holomorphic blocks \( B(q, k) \) and the fusion relations associated to Heegaard decompositions. In [5, 6], it was conjectured that in the flat space limit \( q \to 1 \), the functions \( B_a(q, k) \) degenerate to 2d vortex partition functions. I am in a position to verify this conjecture for these more general blocks. This should help to elucidate the relationship, if any, between two and three dimensional super conformal field theories and the Heegaard splitting.

In addition, localization has also been performed on other geometries in [1, 12, 15, 16]. It is important to determine general factorization conditions for these spaces. This will help us to understand the structure of the partition function and these new representation theoretic invariants \( B(q, k) \). In two dimensions, conformal blocks have lead to many connections between quantum representation theory, the topology of 3-manifolds and algebraic geometry. It is expected that these new invariants will yield similar results in higher dimensions.

Finally, I am interested in continuing to explore what information can be derived about physical theories from integrals of hyperbolic hypergeometric functions. As it stands, the extension of the multidimensional Jordan lemma appearing in my thesis has applications to Mellin-Barnes type integrals, the constructions of classical and basic hypergeometric functions and scattering amplitudes in 4d \( \mathcal{N} = 4 \) super Yang-Mills [7, 11, 22, 23]. I would like to further extend the Jordan lemma to include more complicated classes of functions. In particular, I would like to see how to directly evaluate the partition functions of purely chiral theories.

**Some highlights of thesis work**

**Double sine function and the shift in Chern-Simons level**

One of the central themes of my work is the derivation of information about the spectrum of the twisted Dirac operator from the matrix integral in [1]. Under the action of large gauge transformations with nonzero winding number \( w \) in \( \pi_3(G) \cong \mathbb{Z} \), the eigenvalues of the Dirac operator in three dimensions transform as

\[
\det(\mathcal{D}_A) \overset{w}{\longrightarrow} (-1)^{c_2(\Lambda)} \det(\mathcal{D}_A).
\]

Under these same transformation, the Chern-Simons functional shifts by

\[
\exp(ik \, CS(A)) \overset{w}{\longrightarrow} e^{2\pi i \, w \, k} \exp(ik \, CS(A))
\]

The construction of gauge invariant spinor operators in dimension three requires these two transformation cancel each other:

\[
k - \frac{1}{2} c_2(\Lambda) \in \mathbb{Z}.
\]
Since $c_2(\Lambda)$ may be odd, integrating out highly massive $\mu \gg 1$ matter must enact a compensating shift in the Chern-Simons level. Sensitive one-loop computations like those in [17] show that this formula is given by

$$k \mapsto k - \text{sign}(\mu_j) \frac{1}{2} c_2(\Lambda).$$

We show that this behavior can be seen in the asymptotics of the double sine function. The analytic behavior of the double sine function $s_b(z)$ is given, for Re$(z) \to \pm \infty$, by

$$s_b(z) = |z| \to \infty \exp \left[ \pm \frac{i\pi}{2} \sum_{\beta \in \Delta_j} \left( \frac{\langle \beta, \sigma \rangle}{2\pi} + \mu_j - \frac{i}{2} Q \right)^2 + \frac{1}{12} (b^2 + b^{-2}) + o(1) \right].$$

For a representation $R_j$, the effect of taking $|\mu_j| \to \infty$ is

$$\prod_{\beta \in \Delta_j} s_b \left( \frac{\langle \beta, \sigma \rangle}{2\pi} + \mu_j \right)_{|\mu_j| \to \infty} \exp \left[ - \text{sign} \mu_j \sum_{\beta \in \Delta_j} \frac{\langle \beta, \sigma \rangle^2}{8\pi} + o(1) \right].$$

Here, $\varphi(\mu_j)$ is a quadratic function of $\mu_j$ alone. Note, there are no linear terms in $\sigma$ since for simple Lie groups the weights of any representation sum to zero by Weyl symmetry. Since

$$\sum_{\beta \in \Delta_j} \langle \beta, \sigma \rangle = -c_2(R_j) \cdot \text{Tr}(\sigma),$$

we recover the exact shift in the Chern-Simons level in [9]. In particular, taking $|\mu_j| \to \infty$ shifts the partition function by

$$\lim_{\mu_j \to \infty} e^{\varphi(\mu_j)} Z_{S^3}(k, R_1 \oplus \ldots \oplus R_j) = Z_{S^3}\left( k - \text{sign}(\mu_j) \frac{1}{2} c_2(R_j), R_1 \oplus \ldots \oplus R_{j-1} \right).$$

This result shows that deep information about the spectrum of the Dirac operator can be derived from the asymptotics of the double sine function. It also allows us to evaluate the partition function at arbitrary Chern-Simons level $k$, a result that had not previously been attainable.

**Holomorphic Blocks** As a consequence of [10] the double sine functions $s_b(z)$ decay exponentially in the half-plane Im$(z) < 0$. Additionally, for generic $b \in \mathbb{R}$, they have simple poles on the lattice

$$z_* = -i \, m \, b - i \, n \, b^{-1}, \quad m, n \geq 0.$$

A suitable generalization of the Jordan lemma then allows us to evaluate the partition function $Z_{S^3}(k, \Lambda)$ as an infinite sum of Grothendieck residues of poles a certain set $P \subset \mathbb{C}^r$ of poles. Denoting the integrand of $Z_{S^3}(k, \Lambda)$ by $I_b$,

$$Z_{S^3}(k, \Lambda) = \sum_{\sigma \in P} \text{Res}_{I_b}[I_b]_{\sigma = \sigma*}.$$

The residue sum as presented is a product of basic hypergeometric series in several variables and is almost never computable. However, when the parity condition [9] is satisfied, we show that the sum factorizes into pieces depending on $b$ and pieces depending on $b^{-1}$. In particular, letting $q = e^{2\pi i b^2}$
and \( \bar{q} = e^{2\pi i b^{-2}} \), the infinite sum breaks up into a finite sum over a pairing of basic hypergeometric series \( B_a(\cdot, k) \)

\[
Z_{S^3}(k, \Lambda) = \sum_{a \in I} M_a(k)B_a(q, k)B_a(\bar{q}, k).
\]

Here, \( M_a(k) \) is a function only of \( k \) and the masses \( \mu_j \). The index \( I \) parametrizes sets of \( r \) linearly independent weights of the representations \( \Lambda \) and a lattice polytope associated to each of those sets. Such factorizations were shown to exist in \cite{21} and \cite{24} for the fundamental representation of \( U(N) \). However, there the integrand splits into a product of one dimensional integrals and can be computed by one variable methods. In \cite{4}, I show that this factorization holds for general gauge groups and much more general classes of representations.

It was conjectured in \cite{5} that the blocks \( B_a(q, k) \) are elements of a Hilbert space \( \mathcal{H} \) corresponding to the solid torus \( D^2 \times S^1 \) with metric twisted by \( q \). The decomposition \cite{15} then corresponds to the Heegaard decomposition of the three sphere

\[
S^3_b = D^2 \times S^1 \cup \cdots \cup D^2 \times S^1
\]

\[
Z_{S^3} = \mathcal{H} \otimes \mathcal{H}^*.
\]

For \( U(1) \) these blocks have been shown to construct the partition function on another manifold which can be decomposed into a union of solid tori: \( S^2 \times S^1 \). It is expected that, like conformal block in two dimensions, these blocks are universal representation theoretic objects that may be used to construct the partition function on more general three manifolds \cite{9}. Exploring these claims, in particular now that we have a general expression for the holomorphic blocks, is one of my interests for future research projects.

The higher dimensional Jordan lemma  We briefly recall the higher dimensional Jordan lemma of \cite{23}. For a meromorphic \( r \) form on \( \mathbb{C}^r \) presented as

\[
\omega = \frac{h}{f_1 \cdots f_r} dz_1 \wedge \cdots \wedge dz_r
\]

with \( f_a, h \) holomorphic, define the divisors \( D_a = \{ f_a = 0 \} \). Let \( G_a \subseteq \mathbb{C}, a = 1, \ldots, r \) be domains with piecewise smooth boundaries and let \( g : \mathbb{C}^r \to \mathbb{C}_r \) be a proper map. Define the polyhedral cone \( \Pi = g^{-1}(G_1 \times \cdots \times G_r) \). Denoting the facets of \( \Pi \) as \( \Sigma_a = g^{-1}(G_1 \times \cdots \times \partial G_a \times \cdots \times G_r) \), we say the cone \( \Pi \) is compatible with \( \omega \) if \( D_a \cap \Sigma_a = \emptyset \) for each \( a \), see Figure \cite{2}.

Assume that \( \omega \) dies off asymptotically in \( \Pi \) inside a compatible cone \( \Pi \). If we let

\[
U_a = \Pi - D_a
\]

then \( \omega \in H^0(U_1 \cap \cdots \cap U_r, \Omega^r) \) is a Čech \( r \)-1 co-chain for the sheaf \( \Omega^r \) and covering \( U_a \) of \( U^* = \Pi - \bigcap_a D_a \). By following \( \omega \) through the Dolbeault theorem \( H^{r-1}(U^*, \Omega^r) \cong H^{r-1}_D(U^*) \) composed with the natural mapping \( H^{r-1}_D(U^*) \to H^{2r-1}_D(U^*) \), it can be shown that the integral over the \( r \)-skeleton \( \Gamma = g^{-1}(\partial G_1 \times \cdots \times \partial G_r) \) is given by the sum of residues

\[
\int_{\Gamma} \omega = \sum_{p \in \Pi \cap \bigcap_a D_a} \text{Res} \left[ \omega \right]_{z=p}
\]

where \( \text{Res} \left[ \omega \right]_{z=p} \) is the Grothendieck residue at \( p \).

As has been observed in \cite{11, 14, 22} that for dimension \( r > 2 \) the application of the higher dimensional Jordan lemma is not straightforward. Indeed, I show that if \( \omega \) is singular on more than

\footnote{This is a bit technical but satisfied for reasonable functions decaying exponentially. For a precise definition see \cite{4} or \cite{22}.}
If the domain of integration of $\omega$ is $\Gamma = \mathbb{R}^2$, one can try to find a compatible cone $\Pi$ defined by the positive span of vectors $v_1, v_2 \in \mathbb{R}^2$ as $\Pi = \mathbb{R}^2 + i \text{pos}(v_1, v_2) \subset \mathbb{C}^2$. Pictured are the intersections of linear divisors $f_a(z) = 0$ with $i\mathbb{R}^2$ and the shaded regions denote the intersection $\Pi \cap i\mathbb{R}^2$. The left cone is compatible with $\omega$ since each divisor that intersects $\Pi$ has at least one facet $\Sigma_i$ that it does not intersect. Contrast this with the cone on the right, where one of the divisors intersects both facets $\Sigma_1$ and $\Sigma_2$. For this reason, the cone on the right is not compatible with $\omega$.

$r$ generic polar divisors a compatible polyhedral cone can never be found, even assuming $\omega$ dies off asymptotically in every direction. However, as I show in [4], many forms with large numbers of generic polar divisors can be integrated by using meromorphic partial fraction expansion. I will briefly show this construction in a simple case.

Let $f_a$ be a collection of meromorphic functions with first order zeros $P_a \subset \mathbb{C} - \mathbb{R}$. Assume that the $1/f_a$ dies off exponentially in all directions. The Mittag-Leffler theorem allows us to write each $f_a$ as a sum over the residues

$$
\frac{1}{f_a(z)} = f(0) + \sum_{p \in P_a} \frac{c_p}{z - p} + \frac{c_p}{p}.
$$

(20)

Let $L = \{\lambda_1, \ldots, \lambda_d\}$ be collection of covectors in $((\mathbb{R}^r)^*)$. Assume that $\omega$ can be presented as

$$
\omega = \frac{h(z)}{f_1(\lambda_1(z)) \cdots f_d(\lambda_d(z))} dz_1 \wedge \cdots \wedge dz_r.
$$

(21)

For an index $S \subset \{1, \ldots, d\}$, let $P_S = \{P_{S_1}, \ldots, P_{S_r}\}$ be the ordered set of poles corresponding to each $f_{S_a}$. We can use (20) and partial fraction expansion to write $\omega$ as a sum over $r$-fold products of linear functions

$$
\omega = \sum_{S \subset \{1, \ldots, d\}} \sum_{\{S\} = r} \frac{c_p h(z)}{\prod_{a=1}^{r} (\lambda_a(z) - p_a)} dz_1 \wedge \cdots \wedge dz_r.
$$

(22)

Here, the constant $c_p$ are obtained by taking Grothendieck residues around each pole on both side of (22). They are the constants such that

$$
\text{Res}[\omega]_{z = p} = \text{Res}\left[\frac{c_p h(z)}{\prod_{a=1}^{r} (\lambda_a(z) - p_a)} dz_1 \wedge \cdots \wedge dz_r\right]_{z = p_S}.
$$

(23)

Evaluating the integral over $\omega$ by applying the multidimensional Jordan lemma to (22) yields an interesting result: unlike in the one variable case, many of the residues do not contribute to the pole count. For example, assume that for some $S \subset \{1, \ldots, d\}$, $|S| = r$ and $p \in P_S$, a cone $\Pi$ can be found such that two divisors have no intersection with a single facet, say,

$$
\{\lambda_1(z) = p_1\} \cap \Sigma_1 = \{\lambda_2(z) = p_2\} \cap \Sigma_1 = \emptyset.
$$
Then \( \{ \lambda_1(z) = p_1 \} \cup \{ \lambda_2(z) = p_2 \} \subseteq D_1 \) and so at least one of the other divisors \( D_a, a = 2, \ldots, r \), must be empty. However, if \( D_1 \cap \ldots \cap D_r = \emptyset \) then the residue at \( \bigcap_a \{ \lambda_a(z) = p_a \} \) does not appear in the pole count evaluating the integral of \( \omega \). In \([4]\), I find necessary and sufficient conditions for an intersection of polar divisors to be included in the computation of an integral of a differential form of the shape given in \([21]\). I state the result below.

Assume that \( h(z) \) dominates the asymptotic behavior of \( \omega \) and decays exponentially in the half-plane \( \langle \delta, \text{Im}(z) \rangle \geq 0 \). The nonzero residues have the following form: For the poles \( P_a = \{ f_a = 0 \} \), collect the polar divisors with \( p \in P_a, \text{Im}(p) > 0 \) into \( D_{\lambda_a} = \bigcup_{p \in P_a, \text{Im}(p) > 0} \{ \lambda_a(z) - p = 0 \} \).

Similarly, collect the divisors with \( \text{Im}(p) \leq 0 \) into a divisor \( D_{-\lambda_a} \). Let \( S \) be sets of \( r \) covectors whose positive span contains \( \delta \):

\[
S = \{ \mathcal{L} \subset \{ \pm \lambda_1, \ldots, \pm \lambda_d \} \mid |\mathcal{L}| = r, \delta \in \text{Cone}(\{\lambda_s\}) \}. 
\]

Then the integral over \( \Gamma \) evaluates to

\[
\int_{\Gamma} \omega = \sum_{\mathcal{L} \in S} \sum_{p \in \bigcap_{a=1}^r D_{\mathcal{L},a}} \text{Res}[\omega]_{z=p}. 
\]

We can evaluate integrals of meromorphic forms \( \omega \) exactly in terms the co-vectors \( \lambda_a \), the poles \( P_a \) and a vector \( \delta \) defining an asymptotically safe region. The full proof will appear in \([4]\).

References


