We define $X(t)$ as the solution to the initial value problem, which means for any $a$, $y(t) = X(t)a$ solves

$$\frac{d}{dt}y(t) = Ay(t),$$
$$y(0) = a$$

(1)

(1) Consider the differential equation with the initial condition at $s > 0$.

$$\frac{d}{dt}z(t) = Az(t),$$
$$z(s) = b.$$  

(2)

What is the solution at time $t > s$ in terms of $X$, $t$, $s$, and $b$?

**Solution**

Equation (2) is the same as equation (1) if we translate backwards in time, using $r = t - s$. The solution is then given by the time translation $z(t) = X(t - s)b$. First we make sure that $z(s) = X(0)b = b$ is the right initial condition.

Then check then

$$\frac{d}{dt}z(t) = \frac{d}{dt}X(t - s)b$$
$$= \frac{d}{dr}X(r)b$$
$$= AX(r)b$$
$$= AX(t - s)b$$
$$= Az(t).$$

1
(2) What is the relation of \(X(s)\) and \(X(t)\)? (Give a mathematical formula.)

**Solution**

We can solve (1) by solving \(y(t)\) until \(t = s\) and then solving (2) with \(b = y(s) = X(s)a\). With this choice of \(b\), the solutions are the same \(z(t) = y(t)\). Putting this in terms of the solution operators

\[
z(t) = X(t-s)b = X(t-s)X(s)a
\]

\[
y(t) = X(t)a.
\]

We can conclude that

\[
X(t-s)X(s) = X(t).
\]
(3) (Difficult) Let $q(t) = W(t)c$ be the solution to

$$\frac{d}{dt} y(t) = By(t),$$

$y(0) = c$.

Assume $AB = BA$. What differential equation does $v(t) = W(t)X(t)a$ solve?

Solution

Differentiate $v$ to obtain

$$\frac{d}{dt} v(t) = \frac{d}{dt} W(t)X(t)a$$

$$= \left( \frac{d}{dt} W(t) \right) X(t)a + W(t) \left( \frac{d}{dt} X(t)a \right)$$

$$= BW(t)X(t)a + W(t)AX(t)a.$$

The first step uses a product rule for matrix multiplication. The second step is justified by consider that the matrix $W(t)$ solves the $n \times n$ system of differential equations $\frac{d}{dt} W(t) = BW(t)$.

It is harder to justify that if $A$ and $B$ commute then $A$ and $W(t)$ commute. Once we have established this we can continue with

$$\frac{d}{dt} v(t) = BW(t)X(t)a + AW(t)X(t)a$$

$$= (B + A)v(t).$$

The initial condition is $y(0) = W(0)X(0)a = a$. This is correct only when $AB = BA$.

To justify the missing step that $A$ and $W(t)$ commute, we can show that the commutator, $Z(t) = AW(t) - W(t)A$, is
identically $0$. At $t = 0$, $Z(0) = 0$ because $W(0)$ is the identity matrix. To show this at later times we consider the differential equation that $Z(t)$ solves

$$
\frac{d}{dt} Z(t) = \frac{d}{dt} \left( AW(t) - W(t)A \right) = ABW(t) - BW(t)A.
$$

Using that $A$ and $B$ commute the righthand side is

$$
BAW(t) - BW(t)A = BZ(t).
$$

Then $Z(t) = 0$ is the unique solution to this system of differential equations with initial condition $Z(0) = 0$. Unravelling this,

$$
0 = Z(t) = AW(t) - W(t)A,
$$

which justifies the step we needed to show that $v(t)$ solves (3).