1. ‘Microscopic’ SIR Model

Recall the SIR example from Workshop 2. Let $c$ be the number of contacts per day, $P$ the probability of infection per contact with infected person, $N$ the total number of people, and $\alpha$ is 1 divided by the average days of infection:

\[
\frac{d}{dt}S(t) = -cP \frac{I(t)}{N} S(t),
\]

\[
\frac{d}{dt}I(t) = cP \frac{I(t)}{N} S(t) - \alpha I(t),
\]

\[
\frac{d}{dt}R(t) = \alpha I(t),
\]

with initial conditions

\[
S(0) = N - I_0,
\]

\[
I(0) = I_0,
\]

\[
R(0) = 0.
\]

This system is non-linear. It represents a ‘macroscopic’ model. This means a statistical average of the number of people in each category.

We will now study a ‘microscopic’ model, which describes the state of an individual. An individual can be in one of three states $(S, I, R)$, we let $\rho_S$ be the probability they are susceptible, $\rho_I$ the probability they are infected, and $\rho_R$ the probability they have recovered. We assume that the total number of infected people is constant ($I(t) = I$) and define $\beta = cP \frac{I}{N}$. This is needed so that the model has constant coefficients. If $I(t)$ is given from the solution to the macroscopic model the system will have non-constant coefficients and much more difficult to solve.
The ‘microscopic’ model is:

\[
\frac{d}{dt} \rho_S(t) = -\beta \rho_S(t), \\
\frac{d}{dt} \rho_I(t) = \beta \rho_S(t) - \alpha \rho_I(t), \\
\frac{d}{dt} \rho_R(t) = \alpha \rho_I(t),
\]

with initial conditions (for an initially susceptible student)

\[
\rho_S(0) = 1, \\
\rho_I(0) = 0, \\
\rho_R(0) = 0.
\]

First, let’s vectorize.

\[
y(t) = \begin{bmatrix} \rho_S(t) \\ \rho_I(t) \\ \rho_R(t) \end{bmatrix}.
\]

We define the matrix

\[
A = \begin{bmatrix}
-\beta & 0 & 0 \\
\beta & -\alpha & 0 \\
0 & \alpha & 0
\end{bmatrix}.
\]

The system is now in the form of a linear, homogenous, constant-coefficient system:

\[
\frac{d}{dt} y(t) = Ay(t).
\]

Notice that all columns sum to 0. These matrices are called (infinitesimal) Markov matrices and can represent probability transition rates.

Maybe we want to know the eigenvalues. Since it is lower-triangular, the determinant is the product of diagonal entries.

\[
\det(A - \lambda I) = (-\beta - \lambda)(-\alpha - \lambda)(-\lambda).
\]

The roots are \(-\beta\), \(-\alpha\), and 0. For \(\lambda_1 = 0\) the eigenvector is in the null-space (or kernel) of \(A\) (\(Av_1 = 0\)). It is easy to find because the third column is all zeros, so

\[
v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.
\]

If the initial condition is \(y(0) = v_1\), then the solution is \(y(t) = v_1\).
Let's next do the eigenvalue for $\lambda_2 = -\alpha$. We define $I$ to be the identity matrix,

$$
I = \begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}.
$$

Then we look at the null space of $A - \lambda_2 I$,

$$(A - \lambda_2 I) v_2 = 0,$$

(remember $v_2$ cannot be 0 and this is equivalent to $Av_2 = \lambda_2 v_2$) which is

$$
\begin{bmatrix}
-\beta + \alpha & 0 & 0 \\
\beta & 0 & 0 \\
0 & \alpha & \alpha
\end{bmatrix}v_2 = 0,
$$

Actually this is easy too because the second and third columns are the same. The eigenvector is

$$
v_2 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}.
$$

If we look for solutions of the form $y(t) = a(t)v_2$ then,

$$
\frac{d}{dt}a(t)v_2 = Aa(t)v_2 = -\alpha a(t)v_2,
$$

$$
\frac{d}{dt}a(t) = -\alpha a(t).
$$

So, $a(t) = e^{-\alpha t}$.

Important fact 1: If $v$ is an eigenvector of $A$ with eigenvalue $\lambda$ then $y(t) = e^{\lambda t}v$ is a solution to (3).

What if we started out infected?

$$
y(0) = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = v_2 + v_1.
$$

Because (3) is linear, the solution will be the sum of the solutions starting at $v_1$ and $v_2$. That is

$$
y(t) = e^{-\alpha t}v_2 + v_1 = \begin{bmatrix} 0 \\ e^{-\alpha t} \\ 1 - e^{-\alpha t} \end{bmatrix}.
$$

Important fact 2: A linear combination of solutions is a solution. If $\frac{d}{dt}y_1(t) = Ay_1(t)$ and $\frac{d}{dt}y_2(t) = Ay_2(t)$ then

$$
\frac{d}{dt} (C_1y_1(t) + C_2y_2(t)) = A (C_1y_1(t) + C_2y_2(t))
$$

This follows from linearity of the derivative and matrix multiplication.