Last class: Review workshop. Introduction to Euler’s method (Lebl 1.7)

Today: Second order methods, Improved Euler.

Next topic: Systems (Lebl Chapter 3)

Please print the graphs from the homework and staple them to the rest.
Also, there is now a .tex file for the homework on the webpage as well as the Matlab notes.

From last lecture our goal is to have a numerical approximation to
\[ \frac{d}{dt} y(t) = f(t, y(t)) \]
\[ y(0) = y_0 \]
on \([0, T]\).

In our example,
\[ \frac{d}{dt} y(t) = (1 + t^2) y(t) - y(t)^3 \]
\[ y(0) = 0.5, \]
and the interval is \([0, 2]\).

First review the definitions
- \( h \) is the step size. In matlab \( tval = 0 : h : 2; \)
- \( \tilde{y}_i^h \) is the approximation with step size \( h \) after \( i \) steps, i.e. at time \( t = ih \). (Not raised to a power). Draw a picture. Graph on Matlab.
- Fix \( t = 2 \), define \( \tilde{y}^n = \tilde{y}_i^h \), for \( h = 2^{-n+1} \) and \( i = 2^n \).

We found that the error was linear in \( h \).
\[ |\tilde{y}_i^h - y(t)| \leq \mathcal{C} t h \] for \( t = ih \). In particular,
\[ \frac{|\tilde{y}_{i+1}^h - \tilde{y}_i^h|}{|\tilde{y}_i^h - \tilde{y}_{i-1}^h|} \approx \frac{1}{2} \]
and the geometric series shows that the error is about
\[ |y(2) - \tilde{y}_i^n| \approx |\tilde{y}_i^n - \tilde{y}_{i-1}^n|. \]

Second-order methods. The goal is to have an approximation with \[ |y^h_i - y(t)| \leq \mathcal{C} t h^2. \] This is an improvement by a factor of \( h \). If \( h \) is small this is really good!

Suppose each step for a first order method takes 0.1 s, and 0.5 s for second order method, and in both cases \( \mathcal{C} = t = 1 \). Then to reach an error tolerance of \( 10^{-6} \) the first order method requires \( 10^5 s = 1667 min = 27.8 hr \). The second order method takes 500 s or 8 min.
Taylor expansion one more term
\[ y(t + h) = y(t) + h \frac{dy(t)}{dt} + \frac{h^2}{2} \frac{d^2 y(t)}{dt^2} + C(h)h^3 \]

(4) \[ y(t + h) = y(t) + hf(t, y(t)) + \frac{h^2}{2} \left( \frac{\partial}{\partial t} f(t, y(t)) + f(t, y(t)) \frac{\partial}{\partial y} f(t, y(t)) \right) + C(h)h^3 \]

This uses a multivariable chain rule that you can look forward to learning about in multivariable calculus.

The resulting second order method is:
\[ \hat{y}_0^h = y_0. \]
\[ \hat{y}_{i+1}^h = y_i + hf(t, \hat{y}_i^h) + \frac{h^2}{2} \left( \frac{\partial}{\partial t} f(t, \hat{y}_i^h) + f(t, \hat{y}_i^h) \frac{\partial}{\partial y} f(t, \hat{y}_i^h) \right) \]

for \( t = ih \).

Enter functions (*Lecture6and7Example.m*) and show using *euler_error*.

So halving the timestep with result in \( \frac{1}{4} \) of the error.

Or... be more clever

\[ y(t + h) - y(t) = \int_t^{t+h} f(s, y(s)) ds \]
\[ = \frac{h}{2} \left[ f(t, y(t)) + f(t + h, y(t + h)) \right] + C(h)(h^3) \]
\[ = \frac{h}{2} \left[ f(t, y(t)) + f(t + h, y(t) + hf(t, y(t))) + C(h)(h^2) \right] + C(h)(h^3) \]
\[ = \frac{h}{2} \left[ f(t, y(t)) + f(t + h, y(t) + hf(t, y(t))) \right] + C(h)(h^3) \]

The resulting second order method is called Improved Euler:
\[ \hat{y}_0^h = y_0. \]
\[ \hat{y}_{i+1}^h = y_i + hf(t, \hat{y}_i^h + hf(t, \hat{y}_i^h)) \]

for \( t = ih \).

Introduction to systems.

General first-order system

(5) \[ \frac{d}{dt}y(t) = f(t, y(t)) \]

exactly the same except now \( y \) and \( f \) are vector-valued functions (they will be notated with an arrow on the blackboard). If the dim is \( n \) these are \( n \)-equations for \( n \)-dependent
variables. The Euler (first-order) and improved Euler work exactly the same. Also the existence and uniqueness theorems are the same:

Theorem: Suppose $f$ is continuous and all partial derivatives of $f$ with respect to components of $y$ are uniformly bounded (by a constant independent of $y$). Then there is a unique solution to (5) and it exists for all $t$. 