Today: First order linear equations and integrating factors (Lebl 1.4). Autonomous Equations (Lebl 1.6).

First let’s extend our classification of DE’s.

- **Autonomous Equations**: No independent variable. \( \frac{dy(t)}{dt} = f(y) \). Examples: drag eqn:

  \[ f(v) = g - \frac{\kappa}{m} v^2. \]

  Cooling equation for temperature \( T \) and equilibrium \( T_e \),

  \[ f(T) = k(T_e - T). \]

  May or may not be linear.

- **Constant coefficient linear equations**. Coefficient to \( y \) does not depend on \( t \).

  \[ y' = Ay + h(t) \]

- **Homogeneous linear equations**:

  \[ y' = g(t)y(t) \]

  Important fact: if \( y^1 \) and \( y^2 \) solve a homogeneous equation than so do \( ay^1 + by^2 \) for constants \( a, b \).

  **Quiz**: Does (Constant coefficient linear & Homogeneous) = (Autonomous & Linear)?

  Not quite, (Autonomous & Linear) also includes equations with a constant inhomogeneous term.
Ex:

\[
\frac{d}{dt} y(t) - t y(t) = -t^3
\]

\[
y(0) = 0
\]

First we solve the homogeneous part (any soln works). \( \frac{d}{dt} y^1(t) = ty^1(t) \). Actually we have already done this to get

\[
y^1(t) = e^{\frac{1}{2}t^2}.
\]

To solve for the inhomogeneous part

\[
y(t) = a(t) y^1(t),
\]

\[
\frac{d}{dt} y(t) = \left( \frac{d}{dt} a(t) \right) e^{\frac{1}{2}t^2} + a(t) \frac{d}{dt} y^1(t)
\]

\[
= \left( \frac{d}{dt} a(t) \right) e^{\frac{1}{2}t^2} + a(t) t y^1(t)
\]

\[
\frac{d}{dt} y(t) - t y(t) = \left( \frac{d}{dt} a(t) \right) e^{\frac{1}{2}t^2}
\]

so we want to solve for

\[
\left( \frac{d}{dt} a(t) \right) e^{\frac{1}{2}t^2} = -t^3
\]

\[
\left( \frac{d}{dt} a(t) \right) = -t^3 e^{-\frac{1}{2}t^2}
\]

\[
= t^2 \frac{d}{dt} (e^{-\frac{1}{2}t^2})
\]

\[
= \frac{d}{dt} \left(t^2 e^{-\frac{1}{2}t^2} \right) - 2te^{-\frac{1}{2}t^2}
\]

\[
a(t) = t^2 e^{-\frac{1}{2}t^2} + 2e^{-\frac{1}{2}t^2} + C.
\]

Plug it back in to get

\[
y(t) = t^2 + 2 + Ce^{\frac{1}{2}t^2}
\]

For the initial condition we get \( C = -2 \).

Double check

\[
\frac{d}{dt} y(t) = 2t + Cte^{\frac{1}{2}t^2}
\]

\[
ty - t^3 = t^3 + 2t + tCe^{\frac{1}{2}t^2} - t^3.
\]

General form of first order linear equation:

(1) \[
\frac{d}{dt} y(t) = g(t)y(t) + h(t)
\]
For the first part we solve to get
\[ y^1(t) = C e^{\int_0^t g(s) ds} \]  
(2)

Then we guess 
\[ y(t) = a(t) y^1(t), \]

\[
\frac{d}{dt} y(t) - g(t) y(t) = \left( \frac{d}{dt} a(t) \right) y^1(t) + a(t) \frac{d}{dt} y(t) - g(t) a(t) y^1(t) \\
= \left( \frac{d}{dt} a(t) \right) y^1(t) = h(t)
\]

(3) 
\[ a(t) = \int_0^t \frac{h(s)}{y^1(s)} ds + C \]

To check that \( a(t) y^1(t) \) solves the equation
\[
\left( \frac{d}{dt} a(t) \right) y^1(t) + a(t) \frac{d}{dt} y^1(t) = \frac{h(t)}{y^1(t)} y^1(t) + a(t) g(t) y^1(t) \\
= h(t) + g(t) y(t).
\]

The book uses a slightly different integrating factor \( r \). It does not solve the homogeneous equation but it solves a closely related equation. If you are confused, try to find the equation and the relation between \( r \) from Lebl and \( y^1 \), \( a \) above.

Autonomous equations.

\[ \frac{d}{dt} y(t) = f(y(t)). \]  
(4)

The slope field is constant in \( t \).

We can classify the qualitative behavior of solutions by looking at the critical points, where \( f(y) = 0 \). Example: logistic eqn for population growth. \( \frac{d}{dy} = ky(M - y) \). Draw slope field. Find critical points. Easy, \( y = 0 \) or \( y = M \).

\( y = M \) is stable. If \( y > M \) slightly or \( 0 < y < M \) then \( y(t) \) approaches \( M \) for large \( t \). For \( -y \), the population grows negatively. Thus 0 is unstable. If \( y = 0 \) it says 0 but if \( y > 0 \) or \( y < 0 \) it goes away from 0. This qualitatively describes all the behavior of the solution.