Last class: Nonlinear systems overview, chronic infection model (SIR revisited) (Lebl Ch. 8, Gustafson’s notes on website).

Today: Finish chronic infection model. Start Laplace transform (Lebl Ch. 6)

1. CHRONIC INFECTION MODEL

- Immigration of susceptible people.
- Infection of susceptible people by infected people.
- Death/Emigration (same for susceptible and infected).
- Mortality due to infection.

Draw diagram of connections.

\[
\frac{d}{dt}S(t) = b - dS(t) - \beta \frac{S(t)I(t)}{S(t) + I(t)}
\]

\[
\frac{d}{dt}I(t) = \beta \frac{S(t)I(t)}{S(t) + I(t)} - (d + \alpha)I(t)
\]

.  

Step 0 is to vectorize, let \( \beta = cP \) now

\[
\frac{d}{dt}\begin{bmatrix} S(t) \\ I(t) \end{bmatrix} = \begin{bmatrix} b - dS(t) - \beta \frac{S(t)I(t)}{S(t) + I(t)} \\ \beta \frac{S(t)I(t)}{S(t) + I(t)} - (d + \alpha)I(t) \end{bmatrix}
\]

Compute the equilibrium! Second equation first. Easy solution is \( I = 0 \), then \( S(t) = \frac{b}{d} \). The second one is

\[
0 = b - dS - (\beta - d - \alpha)S = b + (-\beta + \alpha)S \]

\[
S = \frac{b}{\beta - \alpha}
\]

\[
I = \frac{(\beta - d - \alpha)b}{(\alpha + d)(\beta - \alpha)} = \frac{b(R_0 - 1)}{\beta - \alpha}
\]

Only meaningful if both \( S \) and \( I \) are not-negative, \( \beta \geq \alpha + d \).

Note that when \( \beta = \alpha + d \) both the solutions are the same! This is called a bifurcation with bifurcation parameter \( \beta \), the steady state solutions split at the value of \( \beta = \alpha + d \). There will be interesting implications about stability. We will call \( \frac{\beta}{\alpha + d} = R_0 \).
Step 1 Compute Jacobian

\[ D\vec{f} = \begin{bmatrix}
-d - \beta \frac{I}{S+I} + \beta \frac{SI}{(S+I)^2} & -\beta \frac{S}{S+I} + \beta \frac{SI}{(S+I)^2} \\
\beta \frac{I}{S+I} - \beta \frac{SI}{(S+I)^2} & \beta \frac{S}{S+I} - \beta \frac{SI}{(S+I)^2} - d - \alpha
\end{bmatrix} \]

\[ D\vec{f} = \begin{bmatrix}
-d - \beta \frac{I^2}{(S+I)^2} & -\beta \frac{S^2}{(S+I)^2} \\
\beta \frac{I^2}{(S+I)^2} & \beta \frac{S^2}{(S+I)^2} - d - \alpha
\end{bmatrix} \]

If \( I = 0 \) then this is

\[ D\vec{f} = \begin{bmatrix}
-d & -\beta \\
0 & \beta - (d + \alpha)
\end{bmatrix}. \]

Two cases, if \( \beta < d + \alpha \), i.e. \( \eta < 0 \) then this is stable! Otherwise unstable, i.e. an infection might develop. We already have a conclusion that might be important for policy makers.

For the other one plug in the second solution to get something complicated. Notice that

\[ S + I = \frac{\beta b}{(\alpha + d)(\beta - \alpha)} \]

Let

\[ A = \frac{I^2}{(S + I)^2} = \frac{(\beta - d - \alpha)^2}{\beta^2} \]

and

\[ B = \frac{S^2}{(S + I)^2} = \frac{(\alpha + d)^2}{\beta^2}. \]

The Jacobian is

\[ D\vec{f} = \begin{bmatrix}
-d - \beta A & -\beta B \\
\beta A & \beta B - d - \alpha
\end{bmatrix} = \begin{bmatrix} I & II \\
III & IV\end{bmatrix}. \]

At this point we want a better method than just computing eigenvalues. Recall that \( \det(D\vec{f}) = \lambda_1 \lambda_2 \) and \( \text{tr}D\vec{f} = \lambda_1 + \lambda_2. \)

\[
\begin{array}{c|c|c}
\text{det}(D\vec{f}) > 0 & \text{det}(D\vec{f}) < 0 \\
\lambda_1 > 0, \lambda_2 > 0 & \lambda_1 < 0, \lambda_2 < 0 \\
\lambda_1 > 0, \lambda_2 < 0 & \lambda_1 < 0, \lambda_2 > 0 \\
\text{tr}D\vec{f} > 0 & \text{tr}D\vec{f} < 0 \\
\text{Stable!} & \text{Unstable}
\end{array}
\]

For our example we find that \( I < 0, II < 0, III > 0 \) and

\[ IV = \frac{1}{\beta}(\alpha + d - \beta) \]
is negative if $\beta > \alpha + d$ and positive if $\beta < \alpha + d$. So
\[
\det(D\vec{f}) = I(IV) - II(III) > 0
\]
and
\[
\text{tr}D\vec{f} = I + IV < 0
\]
so the equilibrium is stable.

Review the bifurcation diagram.

2. LAPLACE TRANSFORM

We are now going back to linear differential equations like
\[
x'' + x = f(t)
\]
where $f(t)$ is complicated. Some examples we have come across are $e^t$, $\cos(t)$, $te^t$, and piecewise defined functions.

Our goals are the following:

- Systematically compute solutions with a complicated forcing.
- Not have to repeat all the integration by parts over and over again.
- Understand what features of the forcing are important for long-term behavior / other qualitative properties of the solution.
- Gain a new perspective by viewing the problem in the ‘frequency domain’.

The Laplace transform turns a function of time, $f(t)$ for $t > 0$, into a function of $s$, $F(s)$ defined on a new domain, usually $s > s_0$ where $s_0$ depends on $f$.

\[
(2) \quad \mathcal{L}\{f(t)\} = F(s) = \int_0^\infty f(t)e^{-st}dt = \lim_{b \to \infty} \int_0^b f(t)e^{-st}dt
\]
and the domain is where this improper integral exists.

Example, $f(t) = 1$
\[
F(s) = \lim_{b \to \infty} \int_0^b e^{-st}dt = \lim_{b \to \infty} \left[ -\frac{1}{s} e^{-st} \right]_0^b = \lim_{b \to \infty} \left[ -\frac{1}{s} e^{-sb} - \frac{1}{s} e^0 \right].
\]
The limit only exists if $s > 0$ (i.e. $s_0 = 0$). In this case $F(s) = \frac{1}{s}$.