1. Applications of Constant-Coefficient Second Order

1.1. Mass on a Damped Spring (review). The displacement of the spring is \( x(t) \) and the mass is \( m \).

Newton's second law: force equals mass times acceleration. The acceleration is \( x''(t) \).

Hooke's Law: force from spring is negatively proportional to displacement: \(-kx(t)\). Friction is another force that is negatively proportional velocity, \(-\alpha x'(t)\). The resulting 'balance of forces' equation is

\[
xm''(t) + \alpha x'(t) + kx(t) = 0.
\]

The constants satisfy \( k > 0, m > 0 \) and \( \alpha \geq 0 \).

1.2. LCR circuit. Inductor, \( L \), Capacitor, \( C \), and Resistor \( R \). The current is \( I(t) \) (charge/s) and the charge on the capacitor is \( Q(t) \). The voltage is, \( L \frac{dI}{dt} \) for the inductor, \( \frac{Q}{C} \) for the capacitor and \( IR \) for the resistor. Kirchoff's law states that the sum of the voltages should equal the applied voltage (from a battery or outlet). The differential equation is

\[
L \frac{d^2Q}{dt^2} + R \frac{dQ}{dt} + \frac{1}{C}Q(t) = V(t)
\]

where \( V(t) \) is the forcing term, may be alternating (sine) or direct (constant). Despite being a very different physical application, the structure and solutions are very similar to the spring!

1.3. Linear Pendulum. Consider a pendulum that is away from vertical with angle \( \theta(t) \) and has a mass \( m \) at the end of the length \( L \). The force from gravity is \( mg \) in the down
direction, but we only care about the projection on the line perpendicular to the pendulum, which is \(-\sin(\theta)mg\). The velocity is \(L\theta'\), so Newton’s law is
\[
mL\theta''(t) = -mg \sin(\theta(t)).
\]
This equation is nonlinear. If we only care about very small angles, we can approximate \(\sin(\theta) \approx \theta\), and the linear pendulum equation is
\[
(1) \quad mL\theta''(t) + mg\theta(t) = 0.
\]
Again very similar to the spring, we could add friction with a term proportional to \(\theta'(t)\).

2. General Solution for these equations

Let’s just put the equations in the form
\[
x''(t) + bx'(t) + cx(t) = f(t).
\]
Let’s try to write down the general solution operator. We can form a system with \(y_1 = x\) and \(y_2 = x'\) and
\[
\frac{d}{dt} y(t) = \begin{bmatrix} 0 & 1 \\ -c & -b \end{bmatrix} y(t) + \begin{bmatrix} 0 \\ f(t) \end{bmatrix}
\]
The first step is to find the eigenvalues. The characteristic polynomial is
\[
-\lambda(-b - \lambda) + c = 0
\]
\[
\lambda^2 + b\lambda + c = 0
\]
\[
\lambda = \frac{-b \pm \sqrt{b^2 - 4c}}{2}
\]
We assume \(b, c \geq 0\). There are three cases.

(1) If \(b^2 > 4c\), then there are 2 negative real eigenvalues. This is called overdamped. We don’t have to find the eigenvectors because we can work with \(x(t) = c_1e^{\lambda_1 t} + c_2e^{\lambda_2 t}\). You can check that
\[
\begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix}
\]
is actually the eigenvector. The solutions will approach zero with no oscillation.

(2) If \(b^2 = 4c\) this is the critically damped case. In this case the matrix has repeated negative eigenvalues \(\lambda = \frac{-b}{2}\) and is defective. The general solution is \(x(t) = c_1e^{\lambda t} + c_2te^{\lambda t}\). The solutions approach the 0 without oscillating but more slowly.

(3) If \(b^2 < 4c\). Underdamped. Then the eigenvalues are complex. Let’s do some more calculations for this case.
2.1. **Underdamped.** Let $A = \frac{b}{2}$ and $\mu = \frac{\sqrt{4c-b^2}}{2}$. The general solution for $x(t)$ is 

$$x(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

$$= c_1 e^{-At} (\cos(\mu t) + i \sin(\mu t)) + c_2 e^{-At} (\cos(-\mu t) + i \sin(-\mu t))$$

$$= (c_1 + c_2) e^{-At} \cos(\mu t) + i(c_1 - c_2) e^{-At} \sin(\sqrt{\mu} t).$$

To have a real solution, $c_2 = \overline{c_1}$ so $c_1 + c_2$ is real and $c_1 - c_2$ is purely imaginary. We can instead use real constants $a = c_1 + c_2$ and $b = c_1 - c_2$ and

$$x(t) = ae^{-At} \cos(\mu t) + be^{-At} \sin(\mu t).$$

The period of the sin and cos is $\frac{2\pi}{\mu}$ (seconds) and the frequency if $\frac{\mu}{2\pi}$ (Hz). As the damping increases, the period tends to $\infty$ and the solutions approach the critically damped case.