1. DEFECTIVE MATRICES

Consider
\[ A = \begin{bmatrix} \lambda & 0 \\ 1 & \lambda \end{bmatrix}, \]
and we want to be able to solve
\begin{align*}
\frac{d}{dt} y(t) &= Ay(t), \\
y(0) &= y_0
\end{align*}
(1)
for any \( y_0 \). The eigenvalues of \( A \) are \( \lambda_1 = \lambda \) and \( \lambda_2 = \lambda \). To find the eigenvector we look at
\[ A - \lambda I = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}. \]
One eigenvector is
\[ v_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \]
This corresponding solution
\[ y(t) = c_1 e^{\lambda t} v_1. \]
Hopefully, \( y_0 = c_1 v_1 \) and we are done.
If not, let
\[ y(t) = \begin{bmatrix} a(t) \\ b(t) \end{bmatrix}. \]
Equation (1) becomes
\begin{align*}
\frac{d}{dt} a(t) &= \lambda a(t) \\
\frac{d}{dt} b(t) &= a(t) + \lambda b(t).
\end{align*}
For the first equation \( a(t) = c_2 e^{\lambda t} \). Then the second equation is
\[ \frac{d}{dt} b(t) = c_2 e^{\lambda t} + \lambda b(t). \]
We use integrating factors to put this in the form
\[ \frac{d}{dt}(e^{-\lambda t}b(t)) = c_2 e^{\lambda t} e^{-\lambda t} = c_2. \]
So \( b(t) = c_2 t e^{\lambda t} + c_1 e^{\lambda t} \). The general solution is
\[
y(t) = \begin{bmatrix} c_2 e^{\lambda t} \\ c_1 e^{\lambda t} + c_2 t e^{\lambda t} \end{bmatrix} = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} (tv_1 + v_2),
\]
For
\[ v_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}. \]
This is solution works whenever \( v_1 \) is an eigenvector and
\[ A v_2 = \lambda v_2 + v_1. \]
So our method is
(1) Find one eigenvalue.
(2) If there is not a second, then instead we can find \( v_2 \) with \((A - \lambda I)v_2 = v_1\).
(3) Equation (2) gives a general solution.

What if the eigenvalue has multiplicity 3 and only 1 eigenvector? It could be worse, but we will only get higher powers of \( t \) in the solution. For example suppose \( A \) has an eigen value of multiplicity three and only one eigenvectors. Then we can find independent vectors \( v_1, v_2, v_3 \) with \( v_1 \) and eigenvector,
\[
A v_1 = \lambda v_1,
A v_2 = \lambda v_2 + v_1,
A v_3 = \lambda v_3 + v_2.
\]
Then a general solution is
\[
y(t) = c_1 e^{\lambda t} v_1 + c_2 e^{\lambda t} (tv_1 + v_2) + c_3 e^{\lambda t} \left( \frac{t^2}{2} v_1 + tv_2 + v_3 \right).
\]
Check this as an exercise.

2. **Linear Inhomogeneous equations**

General form:
\[
\frac{d}{dt} y(t) = Ay(t) + g(t),
y(0) = y_0.
\]
Suppose we know a particular solution \( y_p(t) \) to
\[
\frac{d}{dt} y_p(t) = Ay_p(t) + g(t),
\]
but \( y_p \) has the wrong initial condition? We can decompose our solution as

\[
y(t) = y_h(t) + y_p(t)
\]

and

\[
\frac{d}{dt}y_h(t) + \frac{d}{dt}y_p(t) = A y_h(t) + A y_p(t) + g(t),
\]

\[
\frac{d}{dt}y(t) = A y(t).
\]

The problem becomes solve the homogeneous problem for \( y_h \) with

\[
y_h(0) = y_0 - y_p(0).
\]

We know how to do that, so all we have to do is find the particular solution.

2.1. **Example with defective matrix.** Two populations with same growth rate. Population 1 eats population 2. But population 2 increasing from a second immigration rate.

\[
\frac{d}{dt}y(t) = \begin{bmatrix}
1 & 0 \\
-1 & 5 & 1
\end{bmatrix} + \begin{bmatrix}
0 \\
t
\end{bmatrix},
\]

\[
y(0) = \begin{bmatrix}
1 \\
1
\end{bmatrix}.
\]

For a particular solution, we can assume population 1 is zero and just solve for

(4) \[
\frac{d}{dt}b(t) = b(t) + t.
\]

The solution by integrating factors is

\[
b(t) = e^t \int e^{-s}sd\bar{s}
\]

\[
= e^t \int \frac{d}{ds}(-e^{-s}t) + e^{-s}d\bar{s}
\]

\[
= e^t (te^{-t} - e^{-t})
\]

\[
= -t - 1.
\]

So

\[
y_p(t) = \begin{bmatrix}
0 \\
-t - 1
\end{bmatrix}
\]

\[
y_p(0) = \begin{bmatrix}
0 \\
-1
\end{bmatrix}
\]

\[
y_h(0) = y_0 - y_p(0) = \begin{bmatrix}
1 \\
2
\end{bmatrix}.
\]

For the general solution to the homogeneous part we need to find \( v_1 \) and \( v_2 \). The eigenvector is still

\[
v_1 = \begin{bmatrix}
0 \\
1
\end{bmatrix}
\]
Next we solve for $\mathbf{A}\mathbf{v}_2 = \mathbf{v}_2 + \mathbf{v}_1$, this is

\begin{equation}
\begin{bmatrix}
0 & 0 \\
-\frac{1}{5} & 0
\end{bmatrix}
\mathbf{v}_2 =
\begin{bmatrix}
0 \\
1
\end{bmatrix}.
\end{equation}

There are more than one solution, but the simplest is

\begin{equation}
\mathbf{v}_2 =
\begin{bmatrix}
-5 \\
0
\end{bmatrix}.
\end{equation}

Now, we express $y_h(0) = c_1\mathbf{v}_1 + c_2\mathbf{v}_2$, we need $c_1 = 2$ and $c_2 = -\frac{1}{5}$. The solution is

\begin{equation}
y(t) = 2e^t\mathbf{v}_1 - \frac{1}{5}(e^t\mathbf{v}_2 + e^t\mathbf{v}_1) + (-t - 1)\mathbf{v}_1,
\end{equation}

\begin{equation}
= \begin{bmatrix}
2e^t - \frac{1}{5}te^t - t - 1
\end{bmatrix}.
\end{equation}