Last Class: LA review. Mass on a Spring. Brief Complex review. Workshop 3


1. Workshop 3 Review

Recall the differential equation from the workshop, \( \frac{d}{dt}y(t) = Ay(t) \), with

\[
A = \begin{pmatrix}
0 & 1 \\
-1 & 0
\end{pmatrix}.
\]

(From mass on a spring with \( m = k = 1 \).

To compute the eigenvalues solve

\[
0 = \det(A - \lambda I) = \lambda^2 + 1.
\]

The roots are \( \lambda_1 = i \) and \( \lambda_2 = -i \). Anything significant? They are conjugate! Recall \( \overline{z} = \text{Re}(z) - \text{Im}(z) \).

To compute the eigenvector for \( \lambda_1 = i \) we compute the null space of \( A - iI \)

\[
A - iI = \begin{pmatrix}
-i & 1 \\
-1 & -i
\end{pmatrix}.
\]

For 2 \( \times \) 2 matrices, if the second column is not all zeros then there will always be an eigenvector with a 1 in the first column. Let's try

\[
\begin{pmatrix}
-i & 1 \\
-1 & -i
\end{pmatrix} \begin{pmatrix} 1 \\ a \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}
\]

The equations are redundant so we only need one of them

\[-i + a = 0 \]

so \( a = i \)

\[
v_1 = \begin{pmatrix} 1 \\ i \end{pmatrix}
\]

What about the second eigenvector?

For real matrices we have a couple facts that makes things a bit easier:

- Fact 1: If \( \lambda \) is an eigenvalue then \( \overline{\lambda} \) is an eigenvalue. (\( \lambda = \overline{\lambda} \) if and only if \( \lambda \) is real.)
- Fact 2: If \( v \) is an eigenvector for \( \lambda \) then \( \overline{v} \) is an eigenvector for \( \overline{\lambda} \). We always have

\[
\overline{Av} = \overline{\lambda}v
\]

\[
A\overline{v} = \overline{\lambda}v
\]

(True only if \( A \) has real entries).
So $v_2 = \nabla_1$ should be the other eigenvector:

$$v_2 = \begin{pmatrix} 1 \\ -i \end{pmatrix}$$

Check anyways:

$$\begin{pmatrix} i & 1 \\ -1 & i \end{pmatrix} \begin{pmatrix} 1 \\ -i \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

All solutions are of the form

$$y(t) = c_1 e^{it} v_1 + c_2 e^{-it} v_2.$$

For the initial condition we want $c_1 = \frac{1}{4}$ and $c_2 = \frac{1}{4}$. The solution is

$$y(t) = \begin{pmatrix} \frac{1}{4} e^{it} + \frac{1}{4} e^{-it} \\ \frac{1}{4} e^{it} + \frac{1}{4} e^{-it} \end{pmatrix}$$

but we want a equation with real numbers so we use Euler’s formula:

$$= \begin{pmatrix} \frac{1}{4} \left( \cos(t) + i \sin(t) \right) + \frac{1}{4} \left( \cos(t) - i \sin(t) \right) \\ \frac{1}{4} \left( \cos(t) + i \sin(t) \right) + \frac{1}{4} \left( \cos(t) - i \sin(t) \right) \end{pmatrix}
= \begin{pmatrix} \frac{1}{2} \cos(t) \\ -\frac{1}{2} \sin(t) \end{pmatrix}$$

Check:

$$\frac{d}{dt} \begin{pmatrix} \frac{1}{2} \cos(t) \\ -\frac{1}{2} \sin(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \sin(t) \\ -\frac{1}{2} \cos(t) \end{pmatrix}$$

$$\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \begin{pmatrix} \frac{1}{2} \cos(t) \\ -\frac{1}{2} \sin(t) \end{pmatrix} = \begin{pmatrix} -\frac{1}{2} \sin(t) \\ -\frac{1}{2} \cos(t) \end{pmatrix}$$

2. Vector Fields for 2D Equations

Draw the vectorfield $Ay$. Also works for non-linear autonomous equations. The stability is determined by the the eigenvalues of $A$. There are basically 3 cases:

- Real eigenvalues. (both positive, mixed both negative)
- Purely imaginary eigen values.
- Complex eigen value with non-zero real part (could be positive or negative)

More on Wednesday with Matlab.