Last week:

- Functions: vertical line test
- One-to-one functions: horizontal line test
- Inverse functions: algebra, graph

Warning about notations

\[ f^{-1}(x) \neq f(x)^{-1} = \frac{1}{f(x)} \]

\[ \text{Inverse function} \quad \text{reciprocal} \]
Figure 1.45

Larger values of $b$ produce greater rates of increase in $b^x$ if $b > 1$.
Figure 1.46

Smaller values of $b$ produce greater rates of decrease in $b^x$ if $0 < b < 1$. 

The graph shows three exponential functions:

- $y = 0.1^x$
- $y = 0.5^x$
- $y = 0.9^x$

As $b$ decreases from 1 to 0, the rate at which $y$ approaches 0 increases.
Figure 1.47

Tangent line has slope 1 at (0, 1).
Exponential functions:

\[ b > 0 \quad f(x) = b^x \quad \text{x any real number} \]

\[ 0 < b < 1 \]

Special value \( b \):

\[ e = 2.718281828459045... \]

\[ 2 < e < 3 \]
Exponent rules

1. \( b^x \cdot b^y = b^{x+y} \)

2. \( \frac{b^x}{b^y} = b^{x-y} \)
   
   In particular, \( \frac{1}{b^x} = b^{-x} \)

3. \( (b^x)^y = b^{x\cdot y} \)

4. \( b^x > 0 \) for all \( x \)

Example

\[ 2^3 \cdot 2^{\frac{1}{2}} = 2^{3\frac{1}{2}} = 2^\frac{7}{2} \]

\[ \frac{2^3}{2^{\frac{1}{2}}} = 2^{\frac{3}{2} - \frac{1}{2}} = 2^1 \]

\[ (2^3)^{\frac{1}{2}} = 2^{\frac{3}{2} \cdot \frac{1}{2}} = 2^{\frac{3}{4}} \]

Fact: \( f(x) = b^x \) is one-to-one.

And so it has an inverse \( f^{-1}(x) = \log_b(x) \)

\[ \log_b(b^x) = x \quad b^{\log_b(x)} = x \]

1. If \( b = e \) we denote \( \log_b(x) = \ln(x) = \log_e(x) = \log(x) = \ln(x) \)

\[ \log_e(e^x) = x \quad e^{\log(x)} = x \]
Figure 1.58

Graphs of $b^x$ and $\log_b x$ are symmetric about $y = x$. 

$y = b^x, \ b > 1$
$y = \log_b x$
$x = x$
$(0, 1)$
$(1, 0)$
The domain of \( f(x) = b^x \) is: any \( x \)
The range is: any \( y > 0 \).

The domain of \( \log_b(x) \) is any \( x > 0 \).
The range of \( \log_b(x) \) is any \( y \).

Logarithmic rules

1. \( \log_b(x \cdot y) = \log_b(x) + \log_b(y) \)

   \[ \log(6) = \log(2) + \log(3) \]

2. \( \log_b \left( \frac{x}{y} \right) = \log_b(x) - \log_b(y) \)

   \[ \log_b \left( \frac{1}{x} \right) = -\log_b(x) \]

   \[ \log_b \left( \frac{3}{2} \right) = \log(3) - \log(2) \]

3. \( \log_b(x^y) = y \cdot \log_b(x) \)

   \[ \log(8) = 3 \cdot \log(2) \]

4. \( \log_b(b) = 1 \)

   \[ \log(e) = 1 \]
Two extra rules:

1. \( b^x = \left( b^{\log(b)} \right)^x = e^{x \cdot \log(b)} \)

2. \( \log_b(x) = \frac{\log(x)}{\log(b)} \)
Figure 1.59

\[ y = \log_b x \]

- \[ y = \log_2 x \]
- \[ y = \ln x \]
- \[ y = \log_5 x \]
- \[ y = \log_{10} x \]

\[ \log_b x \text{ increases on the interval } x > 0 \text{ when } b > 1. \]

\[ \log_b 1 = 0 \text{ for any base } b > 0, b \neq 1. \]
Warm up problem:

Solve $2^x = 3^x$ for $x$. Put your answer in a calculator-ready form.

Solution: 1. $2^0 = 1$, $3^0 = 1$ so $x = 0$ is a solution.

How do we know that this is the only solution?

2. If $2^x = 3^x$ then $\log(2^x) = \log(3^x)$ (by log is a function)

$x \cdot \log(2) = x \cdot \log(3)$

and hence $x \cdot \log(2) - x \cdot \log(3) = 0$

and $x \left( \log(2) - \log(3) \right) = 0$

so $x \cdot \log(\frac{2}{3}) = 0$ and $\log(\frac{2}{3}) = 0$ then $x = 0$. 

3. \(2^x = 3^x\) so \(\frac{2^x}{3^x} = 1\)

\[\frac{b^1}{c^x} = \left(\frac{b}{c}\right)^x\]

hence \((\frac{2}{3})^x = 1\)

so \(\log\left((\frac{2}{3})^x\right) = \log(1)\)

\(x \cdot \log\left(\frac{2}{3}\right) = 0\) \(\Rightarrow x = 0\)
Some business words:

* The revenue is the amount of money \( R \) that we make by selling \( q \) items at a set price \( p \).

\[ R = p \cdot q \]

\( q = 10 \) pens
\( p = 81.50 \)
\( R = 815 \)

* The cost is the amount of money \( C \) that a company spends to make/sell \( q \) items.

\[ C(q) = F + V(q) \]

- Fixed costs
  - Salaries
  - Rent
  - etc.
- Variable costs
  - Material
  - Overtime
  - etc.
The profit is the amount of money $P$ that the company is left with after all products were sold and all costs were paid.

$$P = R - C.$$
Basic Business problem:
- \( p \) - price of each item (in $)
- \( q \) - number of items sold in a week/month/year
- \((p, q)\) is called a data point
- The connection between \( p \) & \( q \) is called law of demand. If \( p \) increases then \( q \) decreases and visa versa.

- Revenue: \( R = p \cdot q \)
- Cost function: \( C(q) = F + V(q) \)

\[ C(q) = F + A \cdot q \]

Profit: \( P = R - C \)

Today: \( V(q) \) is linear.

Today: The demand is linear: \( q = B \cdot p + D \)
Linear function: \( f(x) = Ax + B \)

\( A, B, \) constants

The line passing through \((0, B)\) and having slope \(A\).

\[
\begin{align*}
A + B &= 0 \\
x &= -\frac{B}{A}
\end{align*}
\]
The story:
We were hired by BChalk Inc. They are selling a chalk box for $2 and sell 3,000 boxes a month. Last April, they had a chalk sale (the chalk-fest) and, at a discount of $10 a box, they sold 100 more boxes than other months.

Talking with BChalk's accountant we found that their fixed cost is $3,250 a month and it costs an extra $75 to make a box of chalk, so now their monthly profit is $500 and they would wish to increase it.

1. Find the linear demand equation for a box of chalk. Use the notation \( p \) for the unit price and \( q \) for the monthly demand.

2. Find the monthly cost function, \( C=C(q) \), for producing \( q \) boxes of chalk per month. Note that \( C(q) \) is a linear function.

3. Find the monthly revenue function, \( R=R(q) \). Note that \( R(q) \) is a quadratic function.

4. The break-even points are where Cost equals Revenue; that is, where \( C(q)=R(q) \). Find the break-even points for the product.

5. On the same set of axes, sketch graphs of \( C=C(q) \) and \( R=R(q) \) and use these graphs to help you explain why there are two break-even points.

6. Find the profit function \( P(q)=R(q)-C(q) \). Note that it is a quadratic function.

7. Graph \( P=P(q) \) on the same axes as you sketched the graphs of \( C(q) \) and \( R(q) \). On this graph, indicate the regions of profit \( (P(q)>0) \) and loss \( (P(q)<0) \).

8. How should BChalk Inc. operate in order to maximize the weekly profit \( P=P(q) \)? Use mathematics in your explanation.
\[ q = Bp + D \]

Two data points:

- \((P, q)\) \((2.8, 3,000)\)
- \((1.90, 3,100)\)

\[ \text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3,000 - 3,100}{2.8 - 1.90} = -\frac{100}{0.9} = -1,000 \]

\[ B = \text{Slope} = \frac{+100}{-0.1} = -1,000 \]

Plug in \((2, 3,000)\) into the eq. \(q = -1,000 \cdot p + D\):

\[ 3,000 = -1,000 \cdot 2 + D \Rightarrow D = 5,000 \]

\[ q = -1,000 \cdot p + 5,000 \]
\[ q^* = -1,000p + 5,000 \]

\[ q - 5,000 = -1,000p \]

\[ \frac{-q}{1,000} + 5 = p \]

2. \( F = \$3,250 \)

\[ V(q) = 80.75 - q \]

\[ C(q) = 3,250 + 0.75q \quad \text{[\$/month]} \]

3. \( R = p \cdot q \)

\[ R(q) = p \cdot q = \left( -\frac{q}{1,000} + 5 \right) \cdot q \]

\[ a \quad \text{quadratic function!} \]

4. \( P(q) = R(q) - C(q) \)

\[ = \left( -\frac{q}{1,000} + 5 \right) \cdot q - \left( 3,250 + 0.75q \right) \]

\[ a \quad \text{quadratic function!} \]
\[ P(q) = -\frac{q^2}{1,000} + 4.25q - 3.250 \]

**Demand equations:**

\[ q = -1,000 \cdot p + 5,000 \]

\[ p = \frac{-q}{\frac{1}{1,000}} + 5 \]
\[ A \cdot x^2 + Bx + C = 0 \]

\[
X_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}
\]

\[
X_{\text{max}} = -\frac{B}{2A}
\]

\[
-\frac{q^2}{1000} + 4.25q - 3,250 = 0
\]

\[
\frac{q^2}{1000} - 4.25q + 3,250 = 0
\]

\[
A = 1, \quad B = -4,250, \quad C = 3,250,000
\]

\[
q_{1,2} = \frac{4,250 \pm \sqrt{(-4,250)^2 - 4 \cdot 3,250,000}}{2}
\]

\[
q_{1,2} = \frac{4,250 \pm \sqrt{17,562,500 - 12,400,000}}{2}
\]

\[
q_{1,2} = \frac{4,250 \pm 1,100}{2}
\]

\[
q_{1,2} = \frac{5,350}{2}, \quad \frac{-3,150}{2}
\]

\[
q_{\text{max}} = \frac{q_1 + q_2}{2}
\]

\[
q_{\text{max}} = 2,125
\]
\( q_{\text{max}} = 2,125 \)

\[ P_{\text{max}} = - \frac{q_{\text{max}}}{1,000} + 5 = - \frac{2,125}{1,000} + 5 = 2.875 \text{ $\ell$} \]

\[ P(2,125) = 1,265.625 \text{ $\ell$} \]

The maximal profit is:

\[ = - \frac{2,125^2}{10,000} + 4.25 \cdot 2,125 - 3,250 \]
Introduction to limits:

We threw a stone into the air at a velocity of 30 meters/sec

\[ h(t) = -5 \cdot t^2 + 30 \cdot t \]

How fast is the stone going up after 2 seconds?

\[ h(2) = -5 \cdot 2^2 + 30 \cdot 2 = -20 + 60 = 40 \text{ meters} \]

Average velocity:

\[ \text{Avg } v(t) = \frac{h(2) - h(t)}{2 - t} \quad t \neq 2 \]
<table>
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<th>Interval</th>
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</tr>
<tr>
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<tr>
<td>$[2, 2.000]$</td>
<td>9.995</td>
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We want to say that the velocity of the stone at \( t = 2 \text{ sec.} \) is 10 meters/second.

**Definition:** Suppose the function \( f \) is defined for all \( x \) near \( a \) except possibly at \( a \). If \( f(x) \) is arbitrarily close to \( L \) (as close to \( L \) as we like) for all \( x \) sufficiently close (but not equal) to \( a \), we write
\[
\lim_{x \to a} f(x) = L
\]
and say that the limit of \( f(x) \) as \( x \) approaches \( a \) equals \( L \).

\[
f(t) = \frac{h(t) - h(2)}{2 - t}, \quad L = 10, \quad a = 2
\]
The instantaneous velocity is \( \lim_{t \to 2} \frac{h(t) - h(2)}{2 - t} = 10 \).