Very short answer questions

1. Calculate the derivatives \( f'(x) \) for the following functions:

   (a) 4 marks \( f(x) = e^{\log(x)^2 + \sin(x^2)} \)

   **Answer:** \( e^{\log(x)^2 + \sin(x^2)} \cdot \left( \frac{2 \log(x)}{x} + 2x \cos(x^2) \right) \)

   **Solution:**
   
   \[
   f'(x) = e^{\log(x)^2 + \sin(x^2)} \cdot (\log(x)^2 + \sin(x^2))'
   = e^{\log(x)^2 + \sin(x^2)} \cdot (2 \log(x) \cdot \log(x) + \cos(x^2) \cdot (x^2))'
   = e^{\log(x)^2 + \sin(x^2)} \cdot \left( \frac{2 \log(x)}{x} + 2x \cos(x^2) \right)
   \]

   (b) 4 marks \( f(x) = \frac{(x-4)\sqrt{x^2 + 5x}}{(x^4 - 2) \log(x)} \) **DO NOT SIMPLIFY**

   **Answer:** \( \left( \frac{1}{x-4} + \frac{1}{3} \frac{2x+5}{x^4-2} - \frac{4x^3}{x^4-2} - \frac{1}{x \log(x)} \right) f(x) \)

   **Solution:** Write
   
   \[
   \log(f(x)) = \log \left( \frac{(x-4)\sqrt{x^2 + 5x}}{(x^4 - 2) \log(x)} \right)
   = \log(x-4) + \log \sqrt{x^2 + 5x} - \log(x^4 - 2) - \log(\log(x))
   = \log(x-4) + \frac{1}{3} \log(x^2 + 5x) - \log(x^4 - 2) - \log(\log(x))
   \]

   Deriving both sides yields
   
   \[
   \frac{f'(x)}{f(x)} = \frac{1}{x-4} + \frac{1}{3} \frac{2x+5}{x^2+5x} - \frac{4x^3}{x^4-2} - \frac{1}{x \log(x)}.
   \]

   So
   
   \[
   f'(x) = \left( \frac{1}{x-4} + \frac{1}{3} \frac{2x+5}{x^2+5x} - \frac{4x^3}{x^4-2} - \frac{1}{x \log(x)} \right) f(x).
   \]

2. 6 marks A vending machine stands in an office buildings and sells \( q \) cans of soft drink an hour for the price of \( p \) dollars a can and the demand equation is given by \( p^3q + q^2 = 110 \). Currently the machine sells a can for $1 a can. Use the price elasticity of demand to determine whether the price of a can should be lowered or raised in order to increase their revenue.

   **HINT:** \( \sqrt{441} = 21 \).
\textbf{Solution:} Plugging }p = 1\text{ into the demand equation yields

\[ q + q^2 = 110 \]

The solutions of this equation are

\[ \frac{-1 \pm \sqrt{441}}{2} = -11, +10 \]

The demand must be positive so we take }q = 10.

Differentiating the demand equation \( p^3q + q^2 = 110 \) with respect to }p\text{ yields

\[ 3p^2q + p^3 \frac{dq}{dp} + 2q \frac{dq}{dp} = 0 \]

Plugging in }p = 1\text{ and }q = 10\text{ yields }30 + 21 \frac{dq}{dp} = 0\text{ so that }\frac{dq}{dp} = -\frac{30}{21} = -\frac{10}{7}.\text{ The price elasticity of demand }E = \frac{p}{q} \frac{dq}{dp} = -\frac{1}{7}.\text{ Since }-1 < E < 0\text{ it follows that the price of a can should be increased.}

\[ \text{Long answer questions - You must show your work} \]

3. [6 marks] Consider the curve given by \( y^2 - xy + x^3 = 1 \). Find \( \frac{d^2f}{dx^2}(1) = f''(1) \) assuming that \( y = f(x) \) near \((1, 0)\).

\textbf{Answer: 18}

\textbf{Solution:} We derive the equation once:

\[ 2yy' - xy + 3x^2 = 0 \]

and again:

\[ 2y^2 + 2yy'' - y' - y' - xy'' + 6x = 0. \]

Simplifying, we get:

\[ y'(x) = \frac{y - 3x^2}{2y - x} \]

\[ y''(x) = \frac{2y' - 2y^2 - 6x}{2y - x} \]

Plugging }x = 1\text{ and }y = 0\text{ in the first equation yields

\[ y'(0) = \frac{0 - 3 \cdot 1^2}{2 \cdot 0 - 1} = 3 \]

and plugging }x = 1, y = 0\text{ and }y' = 3\text{ in the second equation yields:

\[ y''(0) = \frac{2 \cdot 3 - 2 \cdot 3^2 - 6 \cdot 1}{2 \cdot 0 - 1} = -18 = 18. \]