Very short answer questions

1. [6 marks] Let \( f(x) \) be a function whose graph is given below. Assume that \( f(x) \) is differentiable in its domain and make a sketch of the graph of \( f'(x) \).

Solution: The graph of \( f'(x) \) is given by the purple curve. The important things in sketch are:

1. The three intercepts of \( f'(x) \), which are the points where \( f(x) \) changes direction between up and down.

2. The interval where \( f'(x) \) is positive or negative, which are the intervals where \( f(x) \) is going up or down respectively.

3. The intervals where \( f'(x) \) is going up or down which are the intervals where \( f(x) \) is "speeding" or "slowing".
2. 6 marks  Show that the equation $2 \log(x) = x^2 - 2$ has a solution.

Solution: Let $f(x) = 2 \log(x) - x^2 + 2$. Note that $f(1) = 2 \log(1) - 1^2 + 2 = 1 > 0$ and that

$$f(e) = 2 \log(e) - e^2 + 2 = 4 - e^2 < 0.$$  

Also, $f(x)$ is continuous on the interval $[1, e]$. Hence, by IVT, there exist $1 < c < e$ such that $f(c) = 0$.

Long answer questions - You must show your work

3. 8 marks  Using only the definition of the derivative, find $f'(3)$, where $f(x) = \sqrt{2x + 3}$ and write the line equation of the tangent to $f(x)$ at $x = 3$. Fully simplify your answer!

Answer: $f'(3) = \frac{1}{3}$,  

$$l(x) = \frac{1}{3}(x - 3) + 3 = \frac{1}{3}x + 2$$
**Solution:** By definition,

\[
f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3}
\]

\[
= \lim_{x \to 3} \frac{\sqrt{2x + 3} - \sqrt{9}}{x - 3}
\]

\[
= \lim_{x \to 3} \frac{\sqrt{2x + 3} - \sqrt{9}}{x - 3} \cdot \frac{\sqrt{2x + 3} + \sqrt{9}}{\sqrt{2x + 3} + \sqrt{9}}
\]

\[
= \lim_{x \to 3} \frac{(\sqrt{2x + 3})^2 - (\sqrt{9})^2}{(x - 3)(\sqrt{2x + 3} + \sqrt{9})}
\]

\[
= \lim_{x \to 3} \frac{2x + 3 - 9}{2x + 3 + \sqrt{9}} = \frac{2}{\sqrt{9} + \sqrt{9}} = \frac{1}{3}
\]

Since \(f(3) = \sqrt{9} = 3\), the tangent is given by

\[
l(x) = f'(3) \cdot (x - 3) + f(3) = \frac{1}{3}(x - 3) + 3.
\]