Simple Interest:

- **PV (Past/Present value)** represents the amount (in dollars) of money loaned (or invested).
- **T** represents the duration, in **years**, of the loan.
- **J** represents the annual simple interest rate of the loan.
- **FV (Future value)** represents the amount of money returned at the end.

\[ FV = PV \cdot (1 + J \cdot T) \]

**Example**

- You loan 1,000$ from the bank for 2 years with an annual simple interest rate of 3%.
- At the end you should return

\[ FV = PV \cdot (1 + J \cdot T) = 1,000 \cdot (1 + 0.03 \cdot 2) = 1,060$ \]
Effective Interest:

- **PV (Past/Present value)** represents the amount (in dollars) of money loaned (or invested).
- **T** represents the duration, in **years**, of the loan.
- **j** represents the annual effective interest rate of the loan.
- **FV (Future value)** represents the amount of money returned at the end.

\[
FV = PV \cdot (1 + j)^T
\]

Here, the interest gained in past years gains interest two in future years.

Example:

- You loan **1,000$** from the bank for **2 years** with an annual effective interest rate of **3%**.
- At the end you should return

\[
FV = PV \cdot (1 + j)^T = 1,000 \cdot (1 + 0.03)^2 = 1,060.9$

\[j > 0.6 \text{ is criminal in Canada}\]
Compounded Interest:

- **PV (Past/Present value)** represents the amount (in dollars) of money loaned (or invested).
- **T** represents the duration, in years, of the loan.
- **i** represents the nominal interest rate of the loan.
- **FV (Future value)** represents the amount of money returned at the end.
- **N** represents the number of compounds per year.

\[ FV = PV \cdot \left(1 + \frac{i}{N}\right)^{NT} \]

Here, the interest gained in each compound gain more interest in future compounds.
Example:

- You loan $1,000 from the bank for 2 years with an annual effective interest rate of 3%. Let's see how much we are getting when compounded N times a year when N varies.

\[ \hat{r} = 0.03 \]

\[ N = 1 : \quad FV = PV \cdot \left(1 + \frac{i}{N}\right)^{NT} = 1,000 \cdot \left(1 + \frac{0.03}{1}\right)^{1 \cdot 2} = 1,060.9\ldots \]$  

\[ N = 2 : \quad FV = PV \cdot \left(1 + \frac{i}{N}\right)^{NT} = 1,000 \cdot \left(1 + \frac{0.03}{2}\right)^{2 \cdot 2} = 1,061.363551\ldots \]$  

\[ N = 4 : \quad FV = PV \cdot \left(1 + \frac{i}{N}\right)^{NT} = 1,000 \cdot \left(1 + \frac{0.03}{4}\right)^{4 \cdot 2} = 1,061.598848\ldots \]$  

\[ N = 12 : \quad FV = PV \cdot \left(1 + \frac{i}{N}\right)^{NT} = 1,000 \cdot \left(1 + \frac{0.03}{12}\right)^{12 \cdot 2} = 1,061.757044\ldots \]$  

\[ N = 365 : \quad FV = PV \cdot \left(1 + \frac{i}{N}\right)^{NT} = 1,000 \cdot \left(1 + \frac{0.03}{365}\right)^{365 \cdot 2} = 1,061.833928\ldots \]  

We have compounded the interest annually, semi-annually, quarterly, monthly and daily. What happens we increase the number compounds?
The answer is that the sequence increases but indefinitely, it approaches a number.

Let's for a second ignore the two year period and consider only the first year.

The effective interest gained with $N$ compounds is

$$
\left(1 + \frac{i}{N}\right)^N
$$

When $N$ grows (to infinity) this number approaches the number

$$
e^i
$$

So, the future value gained with $N$ yearly compounds approaches

$$
FV = PV \cdot e^{i \cdot T}
$$

In our example

$$
FV = PV \cdot e^{i \cdot T} = 1,000 \cdot e^{0.03 \cdot 2} = 1,061.836547\ldots$

Real Interest:

- PV (Past/Present value) represents the amount (in dollars) of money loaned (or invested).
- T represents the duration, in years, of the loan.
- r represents the real interest rate of the loan.
- FV (Future value) represents the amount of money returned at the end.

\[ FV = PV \cdot e^{rT} \]
(Final 2010) How many years will it take for $10,000 to grow to $12,000 if it is invested at 12% annual interest compounded quarterly? You may leave your answer in calculator-ready form.

(Final 2011) You bought a rare stamp collection for 10 million dollars. The auctioneer who sold it to you estimated its value would increase at 12% per year, compounded continuously. If you want to wait until the collection has tripled in value before you sell, how many years will you be waiting?

(Final 2012) You invest $100,000 now at an annual interest rate of 7%, compounded continuously. Your plan is to retire once the rate of growth of your investment is $10,000 per year. In how many years will you retire?

(Final 2013) You borrow 10 thousand dollars from Nick the Shark, who charges you at a fixed rate r that is compounded continuously. If you pay Nick 100 thousand dollars 2 years later, what was the annual rate of interest that he charged? (A calculator-ready form will suffice.)

(Midterm 2016) An investment of $35,000 grew to $37,000 over 18 months. What is the effective interest rate?
Final 2010:

\[ FV = PV \left(1 + \frac{i}{N}\right)^{NT} \]

**Data:**

- \[ PV = 10,000 \text{ } \$ \]
- \[ FV = 12,000 \text{ } \$ \]
- \[ i = 0.12 \]
- \[ N = 4 \text{ (quarterly!)} \]

**Solve for \( T \):**

\[
10,000 = 12,000 \left(1 + \frac{0.12}{4}\right)^{4T}
\]

\[
\frac{10,000}{12,000} = \left(1 + \frac{0.12}{4}\right)^{4T}
\]

\[
\frac{5}{6} = \left(1 + \frac{0.12}{4}\right)^{4T}
\]

\[
\log \left(\frac{5}{6}\right) = 4T \cdot \log (1.03)
\]

\[
\log (\frac{5}{6}) = \log \left[(1 + 0.03)^4\right] = 4T \cdot \log (1.03)
\]

\[
T = \frac{\log (1.2)}{4 \log (1.03)} \approx 1.54
\]

Write relevant formulas in the test.

- **daily** \( N = 365 \)
- **monthly** \( N = 12 \)
- **quarterly** \( N = 4 \)
- **semi-annually** \( N = 2 \)
- **annually** \( N = 1 \)

1. \( T \) is always measured in years.

2. Calculators ready answer is enough. You are
Final 2011: \[ FV = PV e^{rT} \]

**Data:**
\[ r = 0.12 \]
\[ PV = 10,000,000 \quad \text{(irrelevant)} \]
\[ FV = 3 \cdot PV \]

\[ 3 \cdot PV = DV e^{0.12 \cdot T} \]
\[ e^{0.12 \cdot T} = 3 \]
\[ 0.12 \cdot T = \log(3) \quad \Rightarrow \quad T = \frac{\log(3)}{0.12} \approx 9.155 \ldots \text{ years} \]

Final 2012:
\[ PV = 100,000 \]
\[ r = 0.04 \]
\[ FV, T = ? \]

Let's try again.
We want to find FV so that
\[ FV + 10,000 = FV \cdot e^{0.12} \]

(earn 10,000 $ every year).

\[ FV \cdot (e^{0.12} - 1) = 10,000 \]
\[ FV = 10,000 \]
Final 2013: \[ FV = PV e^{rt} \]

Data:
- \( FV = 100,000 \)
- \( T = 2 \) (years)
- \( PV = 10,000 \)

\[ 100,000 = 10,000 \cdot e^{2r} \]
\[ 10 = e^{2r} \]
\[ 2r = \log_{10} 10 \]
\[ r = \frac{\log_{10} 10}{2} = 1.15 \ldots \]  

(Way over the legal 6%)

Midterm 2016:
\[ FV = PV \left(1 + \frac{r}{n}\right)^{nt} \]

\[ 37,000 = 35,000 \left(1 + \frac{1/3}{35}\right)^{18} \]
\[ \frac{18}{35} = \frac{2}{5} \]

\[ \left(\frac{2/3}{35}\right)^{18/3} = 1 + j \]
\[ j = \left(\frac{2}{35}\right)^{18/3} - 1 \approx 0.0372 \ldots \]

What is the real interest rate?
\[ FV = PV e^{rt} \]
\[ rT = \log \left(\frac{37/35}{10/100}\right) \approx 0.024 \ldots \]