Last Week:
- Concavity
- Limits involving infinity
- Curve Sketching
- Optimization

Today:
- More Optimization

Tomorrow:
- L'Hôpital's rule
- Linear approximations
- Taylor polynomials.

Wednesday & Thursday: Review.
What shall we do?

Office Hours next week: Monday & Tuesday
2-4 pm.
LSK 200 or 301
Question 12. A truck enters a road shaped as a quarter of a circle with radius 20 kilometres. The driver wants to reach a construction site at point A off the road located exactly between the centre of the circle and the end of the road. He can drive some distance along the road at 60 km/hr and then get off the road at 30 km/hr. How many kilometres should the driver drive on the road, in order to minimize its travel time?

Target function: \( T_{\text{min}} \)

\[
T = \frac{a}{60} + \frac{b}{30}
\]

**Constraints:**

\( 0 \leq a \leq \frac{\pi}{2} \cdot R = \pi \cdot 10 \)

\( 0 \leq b \leq 20 \)

\( a = \Theta \cdot R = 20 \cdot \Theta \)

\( 0 \leq \Theta \leq \frac{\pi}{2} \)

\[
b^2 = 10^2 + 20^2 - 2 \cdot 10 \cdot 20 \cdot \cos \left( \frac{\pi}{2} - \Theta \right) \implies b = 10 \sqrt{5 - 4 \sin^2 \Theta}
\]

\[
= 10^2 \left[ 5 - 4 \frac{\cos \left( \frac{\pi}{2} - \Theta \right)}{\sin \Theta} \right] = 10^2 \left[ 5 - 4 \sin^2 \Theta \right]
\]

\[
T(\Theta) = \frac{20 \cdot \Theta}{60} + \frac{10 \sqrt{5 - 4 \sin^2 \Theta}}{30} = \frac{1}{3} \Theta + \frac{1}{3} \sqrt{5 - 4 \sin^2 \Theta}
\]
Use CIR:

\[ T'(\theta) = \frac{1}{3} + \frac{1}{3} \cdot \frac{-4 \cos \theta}{2 \sqrt{5 - 4 \sin^2 \theta}} = 0 \]

\[ 2\sqrt{5 - 4 \sin^2 \theta} - 4 \cos \theta = 0 \]
\[ \sqrt{5 - 4 \sin^2 \theta} = 2 \cos \theta \]
\[ 5 - 4 \sin^2 \theta = 4 \cos^2 \theta = 4(1 - \sin^2 \theta) \]
\[ 4 \sin^2 \theta - 4 \sin \theta + 1 = 0 \]
\[ \sin \theta = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{6}. \]

\[ T(0) \approx 0.745 \quad \text{[chr]} \]
\[ T(\frac{\pi}{6}) \approx 0.751 \quad \text{[chr]} \]
\[ T(\frac{\pi}{2}) \approx 0.856 \quad \text{[chr]} \]

Conclusion: The driver should go straight to point A.
Question 11. A prince is drowning 20 meters off a strait coast. A princess sees him and wants to save him. She is standing 30 meters off the coast line and 50 meters to the right. She can run at the speed of 3 meters/second and swim at the speed of 1 meter/second. What is the course in which she should run and swim to get to the prince as fast as she can?

Target function:

\[ T = \frac{\sqrt{30^2 + x^2}}{3} + \frac{\sqrt{20^2 + (50-x)^2}}{1} \text{ km} \]

Constraints: \(0 \leq x \leq 50\)

Optimal value:

\[ \frac{dT}{dx} = \frac{2x}{2\sqrt{30^2 + x^2}} + \frac{2(50-x)}{2\sqrt{20^2 + (50-x)^2}} \]

\[ = \frac{x}{\sqrt{30^2 + x^2}} - \frac{50-x}{\sqrt{20^2 + (50-x)^2}} \]

\[ = \frac{\sin \Theta_1}{3} - \frac{\sin \Theta_2}{1} \]

\(\Theta_1 = \frac{3 \text{ m}}{\text{sec}}\)

\(\Theta_2 = \frac{1 \text{ m}}{\text{sec}}\)

\(\frac{\sin \Theta_1}{\Theta_1} - \frac{\sin \Theta_2}{\Theta_2} = 0 \Rightarrow \Theta_1 = \Theta_2\)

Aside: \(\frac{dT}{dx} = 0 \Rightarrow x = 44.35\) m, 56.18 m

Snell's Law.
L'Hôpital's Rule

**Theorem:** Let \( f \) and \( g \) be diff on an open interval \( I \) except possibly at a point \( c \) contained in \( I \). If
\[
\lim_{x \to c} f(x) = \lim_{x \to c} g(x) = 0
\]
and \( g'(x) \neq 0 \) for all \( x \) in \( I \) and \( \lim_{x \to c} \frac{f'(x)}{g'(x)} \) exists,

then
\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}.
\]

**Remark:** Also true for one-sided limits.

* Also true for \( \lim_{x \to c} f(x) = \lim_{x \to c} g(x) = \infty \)

**Why should this make sense?** \( f(x) \), \( g(x) \) are cont.
\[
f(x) = f(c) + f'(c)(x - c) \text{ near } c \text{ at } c.
\]
\[
g(x) = g(c) + g'(c)(x - c)
\]
\[
\frac{f(x) - f(c)}{g(x) - g(c)} = \frac{f'(c)(x - c)}{g'(c)(x - c)} = \frac{f'(c)}{g'(c)}
\]

\[
\lim_{x \to c} \frac{f(x)}{g(x)} = \frac{f'(c)}{g'(c)}.
\]
Yesterday:
Optimization

Today:
- L'Hôpital's Rule
- Linear approximation (+errors in them)
- Taylor polynomials

L'Hôpital's rule:
Let $f$ and $g$ be diff at an open interval $I$ except possibly a point $c \in I$.
If:
* $\lim_{x \to c^\pm} |f(x)| = \lim_{x \to c^\pm} |g(x)| = 0$ (or $\pm \infty$)
* $g'(x) \neq 0$ for all $x \neq c$ in $I$
* $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exists.
Then $\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$. 
Good Examples:

* \( \lim_{{x \to 0}} \frac{e^x - 1}{x^2 + x} \) \( \equiv \lim_{{x \to 0}} \frac{e^x}{2x + 1} = \frac{e^0}{2 \cdot 0 + 1} = 1 \)

* \( \lim_{{x \to 0}} \frac{\sin^2 x}{\cos x - 1} \) \( \equiv \lim_{{x \to 0}} \frac{2 \sin x \cdot \cos x}{-\sin x} = \lim_{{x \to 0}} \frac{2 \cos x}{-1} = \frac{2 \cdot 1}{-1} = -2 \)

* \( \lim_{{x \to 0^+}} x \cdot \ln x = \lim_{{x \to 0^+}} \frac{\ln x}{(1/x)} \equiv \lim_{{x \to 0^+}} \frac{(1/x)}{(-1/x^2)} = \lim_{{x \to 0^+}} (-x) = 0 \)

* \( \lim_{{x \to 0}} \frac{2 \sin x - \sin(2x)}{x - \sin x} \) \( \equiv \lim_{{x \to 0}} \frac{2 \cos x - 2 \cos(2x)}{1 - \cos x} \)

\( \equiv \lim_{{x \to 0}} \frac{-2 \sin x + 4 \sin(2x)}{\sin x} \) \( \equiv \lim_{{x \to 0}} \frac{-2 \sin x + 8 \cos(2x)}{\cos x} = \frac{-2 \cdot 0 + 8}{1} = 6 \)
Bad Examples:

* \( \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} \) does not equal \( \lim_{x \to \infty} \frac{e^x - e^{-x}}{e^x + e^{-x}} \).

L'Hopital's rule doesn't help.

* \( \lim_{x \to \infty} \frac{e^x + e^{-x}}{e^x - e^{-x}} = \lim_{x \to \infty} \frac{1 + e^{-2x}}{1 - e^{-2x}} = \frac{1 + 0}{1 - 0} = 1 \)

* \( \lim_{x \to \infty} \frac{x + \sin x}{x} \) does not equal \( \lim_{x \to \infty} \frac{1 + \cos x}{1} = 1 + \lim_{x \to \infty} (\cos x) \) which is undefined.

Aside: \( \lim_{x \to \infty} \frac{x + \sin x}{x} = \lim_{x \to \infty} \left( 1 + \frac{\sin x}{x} \right) = 1 \)
Linear Approximations:

Example: Approximate $\sqrt{73}$ as a simple fraction without using a calculator.

Solution: Let's use the tangent of $f(x) = \sqrt{x}$ at $x = 8$ or 9.

$$l(x+h) = f(x) + f'(x)h$$

$$\frac{f(x+h) - f(x)}{h} \approx f'(x)$$

$8^2 = 64 < 73 < 8^2 = 64$ 

$f(64) = 8$

$f'(64) = \frac{1}{2\cdot 8} = \frac{1}{16}$

$l(x+h) = l(x) + f'(x)h$

$73 - 64 = 9$

$f(73) \approx \frac{9}{16} + 8 = \frac{137}{16}$

$f(81) = 9$

$f'(81) = \frac{1}{2\cdot 9} = \frac{1}{18}$

$l(h) = \frac{h}{18} + 9$

$73 - 81 = -8$

$l(9+h) = \frac{-8}{18} + 9 = \frac{73}{9}$
Both \( \frac{13T}{16} \) and \( \frac{7T}{9} \) are overestimates of \( \sqrt{3} \). \( f''(x) = -\frac{1}{4x^{3/2}} < 0 \) so \( f \) is \( \text{CD} \).

Since \( \frac{7T}{9} < \frac{13T}{16} \), \( \frac{7T}{9} \) should be a better estimate.

\( \sqrt{3} \approx 1.732 \), \( \frac{7T}{9} \approx 8.555 \ldots \), \( \frac{13T}{16} \approx 8.562 \ldots \).
Concave down → Tangents are above the graph
   → linear approx. are over-estimates.

Concave Up → Tangents are below the graph
   → linear approx. are under-estimates.
More examples:

1. \( \sqrt[3]{7} \approx ? \)

Let's use \( f(x) = \sqrt[3]{x} \). \( f(7) \approx ? \)

We'll use the tangent of \( f(x) \) at 8.

\[ f(8) = \sqrt[3]{8} = 2. \]

\[ f'(x) = \left( x^{1/3} \right)' = \frac{1}{3} x^{-2/3} \]

\[ f'(8) = \frac{1}{3} \cdot 8^{-2/3} = \frac{1}{3} \cdot 2^2 = \frac{1}{12}. \]

\[ l(8+h) = \frac{h}{12} + 2. \]

\[ +h = -1 \quad \text{so} \quad \sqrt[3]{7} \approx l(7) = \frac{-1}{12} + 2 = \frac{23}{12}. \]

\( f(x) \) is CD

So \( \frac{23}{12} \) is an overestimate of \( \sqrt[3]{7} \).
Say that \( y = f(x) \) is given by \( x^2 + y^3 - 2xy = 0 \) near \((1,1)\), approximate \( f(1.02) \).

Set: Let's write the tangent of \( f(x) \) at 1.

\[ f(1) = 1. \]

\[ f'(1) = ? \]

\[ 2x + 3y^2 \cdot y' - 2y - 2xy' = 0 \]

Plug \( x = 1, y = 1, y' = f'(1) \) to get:

\[ 2 + 3f'(1) - 2 - 2f'(1) = 0 \]

\[ f'(1) = 0. \]

\[ l(x) = 1 \]

\[ f(1.02) \approx 1. \]
Taylor Polynomials:

Example: Approximate $\log(2)$ without using a calculator.

Sol:

Use:

$f(x) = \log(x)$.

$f(1) = \log(1) = 0$

$f'(x) = \frac{1}{x}$

$f'(1) = \frac{1}{1} = 1$

$f'(e) = \frac{1}{e}$

$\log(1+h) = h + O = h$

$\log(2) \approx \log(1) + 1 = 1$

$\log(2) \approx 0.69$

$\log(x)$ is concave down so it's an underestimate.

$l(x) = f'(x_0) \cdot (x-x_0) + f(x_0) \quad \text{tangent of } f(x) \text{ at } x_0.$
How to improve that:

\[ \log(x) = -\log\left(\frac{1}{x}\right) \approx -\log\left(\frac{1}{2}\right) = \frac{1}{2}. \]

Let's find something better than a linear approx.
Let's find a quadratic function which approx. \( p(x) = \log(x) \) near \( x = 1 \).

\[ T_2(x) = A + Bx + Cx^2 \]

\[
\begin{align*}
T_2(1) &= f(1) & \text{← Same value} \\
T_2'(1) &= f'(1) & \text{← Same tangent} \\
T_2''(1) &= f''(1) & \\
\end{align*}
\]

Better: \( T_2(x) = D + E \cdot (x-1) + F \cdot (x-1)^2 \)

\[
\begin{align*}
D &= 0 & A + B + C &= 0 \\
E &= 1 & B + 2E &= 1 \\
2F &= -1 & 2-C &= -1 \\
\end{align*}
\]
\[ D = T_2(1) = f'(1) = \log(1) = 0 \]

\[ T_2'(x) = E + \alpha_f(x-1) \]

\[ E = T_2'(1) = f'(1) = \frac{1}{1} = 1 \]

\[ f'(x) = \frac{1}{x} \]

\[ T_2''(x) = \alpha_f \]

\[ \alpha_f = T_2''(1) = f''(1) = -1 \]

\[ f''(x) = -\frac{1}{x^2} \]

\[ \Rightarrow \quad D = 0, \quad E = 1, \quad \alpha_f = -\frac{1}{2} \]

\[ \overline{T_2(x)} = (x-1) - \frac{1}{2} (x-1)^2 \]

\[ \log(2) \approx \overline{T_2(2)} = \frac{1}{3} \]
Let's try a cubic approximation:

\[ T_3(x) = D + E(x-1) + F(x-1)^2 + G(x-1)^3 \]

\[ T_3(1) = f(1) \quad D = 0 \]
\[ T_3'(1) = f'(1) \quad E = 1 \]
\[ T_3''(1) = f''(1) \quad 2F = -1 \]
\[ T_3'''(1) = f'''(1) \quad 6G = 2 \]

\[ T_3'(x) = E + 2F(x-1) + 3G(x-1)^2 \]
\[ T_3''(x) = 2F + 6G(x-1) \]
\[ T_3'''(x) = 6G \]
\[ f'''(x) = \frac{a}{x^3} \]

\[ T_3(x) = (x-1) - \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3. \]

\[ \log(6) \approx T_3(2) = \frac{5}{2} \approx 0.83. \]
The $n$th degree Taylor polynomial of $f(x)$ at $x_0$ is the polynomial

$$T_n(x) = a_0 + a_1(x-x_0) + a_2(x-x_0)^2 + \ldots + a_n(x-x_0)^n$$

5.1. $T_n(x_0) = f(x_0)$

$$T_n'(x_0) = f'(x_0)$$

$$T_n^{(n)}(x_0) = f^{(n)}(x_0)$$

$$a_0 = f(x_0)$$

$$a_1 = f'(x_0)$$

$$2a_2 = f''(x_0)$$

$$6a_3 = f'''(x_0)$$

$$-n!a_n = f^{(n)}(x_0)$$

$$k!a_k = f^{(k)}(x_0)$$

So:

$$a_k = \frac{f^{(k)}(x_0)}{k!}, \text{ where}$$

$k!$ is defined by: $0! = 1$, $1! = 1$, $k! = (k-1)! \cdot k$

So:

$$0! = 1, \ 1! = 1, \ 2! = 2, \ 3! = 6, \ 4! = 24, \ 5! = 120,$$
Examples of Taylor polynomials with $x_0 = 0$

(MacLaurin polynomials)

1. $f(x) = e^x$
   - $f(x) = e^x$
   - $f^{(n)}(x) = e^x$
   - $f^{(n)}(0) = 1$
   - $T_0(x) = 1$
   - $T_1(x) = 1 + x$
   - $T_2(x) = 1 + x + \frac{1}{2} \cdot x^2$
   - $T_3(x) = 1 + x + \frac{1}{2} \cdot x^2 + \frac{1}{6} \cdot x^3$
   - $T_4(x) = 1 + x + \frac{1}{2} \cdot x^2 + \frac{1}{6} \cdot x^3 + \frac{1}{24} \cdot x^4$

2. $f(x) = \log(1 + x)$
   - $T_0(x) = 0$
   - $T_1(x) = x$
   - $T_2(x) = x - \frac{1}{2} \cdot x^2$
   - $T_3(x) = x - \frac{1}{2} \cdot x^2 + \frac{1}{3} \cdot x^3$
   - $T_4(x) = x - \frac{1}{2} \cdot x^2 + \frac{1}{3} \cdot x^3 - \frac{1}{4} \cdot x^4$
3. \( f(x) = \cos x \quad T_4(x) = 1 - \frac{1}{2} x^2 + \frac{1}{24} x^4 \)

4. \( f(x) = \sin x \quad T_4(x) = x - \frac{x^3}{6} \)

5. \( f(x) = \frac{1}{1-x} \)

\[ T_n(x) = 1 + x + x^2 + x^3 + \ldots + x^n. \]

Example: Find the 3rd Maclaurin poly. of \( f(x) = e^{\cos x} \).

Solv. \( f(0) = e^0 \cos(0) = 1 \)

\[ f'(x) = e^{x \cos x} - e^{x \sin x} \quad f'(0) = 1 \]

\[ f''(x) = (e^{x \cos x} - e^{x \sin x}) - (e^{x \sin x} + e^{x \cos x}) \]

\[ = -2e^{x \sin x} \quad f''(0) = 0 \]

\[ f'''(x) = -2(e^{x \sin x} + e^{x \cos x}) \quad f'''(0) = -2 \]

\[ T_3(x) = 1 + f'(0)x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 = 1 + x - \frac{1}{3} x^3. \]
Alternative way: \( f(x) = e^x \cos x \)

\( e^x: \quad P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} \)

\( \cos x: \quad Q_3(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} \)

\[ P_3(x) Q_3(x) = (1 + x + \frac{x^2}{2} + \frac{x^3}{6})(1 - \frac{x^2}{2}) + \]

\[ = 1 + x - \frac{1}{3} x^2 + \frac{1}{2} x^2 - \frac{1}{6} x^3 + \frac{1}{6} x^3 - \frac{x^4}{4} + \frac{x^5}{12} \]

\[ = 1 + x - \frac{1}{3} x^3 \]

\[ P_3(x) = 1 + x - \frac{1}{3} x^3 \]
Yesterday
- L'Hôpital's rule
- linear approximations
- Taylor polynomials

Today:
- Taylor polynomials - more examples
- Error bounds for linear approx.
- Review

Taylor polynomials: The $n^{th}$ Taylor polynomial of $f(x)$ at $a$ is:

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \ldots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

If $a=0$, $T_n(x)$ is called the $n^{th}$ Maclaurin poly.
Example: Find the 3rd Maclaurin poly. of \( f(x) = e^{x^2} \).

Solution: \( T_3(x) = f(0) + f'(0) \cdot x + \frac{f''(0)}{2!} x^2 + \frac{f'''(0)}{3!} x^3 \)

\( f(0) = e^0 = 1 \)
\( f'(0) = 2 \cdot 0 = 0 \)
\( f''(0) = 2 \)
\( f'''(0) = 0 \)

\( f'(x) = e^{x^2} \cdot 2x \)
\( f''(x) = e^{x^2} (2x)^2 + 2e^{x^2} \)
\( = e^{x^2} (4x^2 + 2) \)
\( f'''(x) = e^{x^2} \cdot 2x \cdot (4x^2 + 2) + 2e^{x^2} \cdot 8x \)

\( T_3(x) = 1 + x^2 + 0 \cdot x^3 \)

Alternative Solution:
\( e^{x^2} \): \( P_3(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \)

\( P_3(x^2) = 1 + x^2 + \left( \frac{x^4}{2} + \frac{x^6}{6} \right) + = 1 + x^2 + \)

\( T_3(x) = 1 + x^2 \)
Ex: The 7th Mac. poly of \( f(x) = e^{3\sin x} \) is
\[ T_7(x) = 1 + x + \frac{x^2}{2} - \frac{x^4}{8} - \frac{x^5}{15} + \frac{x^6}{240} + \frac{x^7}{340}. \]
Find \( f^{(6)}(0) \).

Sol:
\[ T_7(x) = f(0) + f'(0) x + \frac{f''(0)}{2!} x^2 + \ldots + \frac{f^{(6)}(0)}{6!} x^6 + \frac{f^{(7)}(0)}{7!} x^7. \]
So
\[ \frac{f^{(6)}(0)}{6!} = -\frac{1}{240}. \]

\[ f^{(6)}(0) = -\frac{6!}{240} = -\frac{720}{240} = -3. \]

Ex: \( x^3 - 3x^2 + 2 = 0 \)

Integral coefficients + leading coeff. is 1
\( \Rightarrow \) good chances that divisors of const. coeff. are roots.

Divisors of +2: ±1, ±2. \( \Rightarrow \) +1 is a root.
Let's write the 3rd Taylor polynomial of \( f(x) = x^3 - 3x^2 + 2 \) at 1.

\[
\begin{align*}
    f(x) &= x^3 - 3x^2 + 2 \\
    f'(x) &= 3x^2 - 6x \\
    f''(x) &= 6x - 6 \\
    f'''(x) &= 6 \\
    f(1) &= 0 \\
    f'(1) &= -3 \\
    f''(1) &= 0 \\
    f'''(1) &= 6
\end{align*}
\]

\[
x^3 - 3x^2 + 2 = T_3(x) = -3(x-1) + \frac{6}{6} (x-1)^3 = (x-1) \left[ (x-1)^2 - 3 \right]
\]

\[
= (x-1) \left[ x^2 - 2x - 2 \right]
\]
Errors Bounds for Linear Approximations

\[ T_1(x) = \ell(x) = f(a) + f'(a)(x-a) \]

Tangent at \( a \) of \( f(x) \)

\[ T_1(x) = \ell(x) \text{ is near } f(x) \text{ when } x \text{ is near } a. \]

But how good of an approx. is it?

Namely, can we estimate \( |f(b) - \ell(b)| \)?

Can we say what is the worst case error?

**Theorem (Last one):** Let \( f \) be twice diff. between \( a \) and \( b \). Let \( M \geq 0 \) be a constant such that

\[ |f''(x)| \leq M \text{ for any } x \text{ between } a \text{ and } b. \]

Then

\[ |f(b) - \ell(b)| \leq \frac{M}{2} (b-a)^2. \]

Note that we usually know whether \( f(b) \geq \ell(b) \)

(Using Concavity)

or \( f(b) < \ell(b) \).
Why would that be true/What does it mean?

We know that near $a$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2}(x-a)^2$$

So

$$f(b) = f(a) + f'(a)(b-a) + \frac{f''(a)}{2}(b-a)^2$$

Therefore

$$|f(b) - l(b)| \leq \frac{M}{2} (b-a)^2$$

where $M = \max|f''(x)|$ on $[a, b]$. 

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The graph shows the function $f(x)$, its Taylor polynomial $l(x)$, and the error term $|f(b) - l(b)|$.
Example (Final 2011)

\[ \log(0.9) \approx ? \] How good is this approx.?

**Sol:** \[ l(x) = x - 1 \]

\[ \log(0.9) \approx l(0.9) = 0.9 - 1 = -0.1 \]

The tangent of \( f(x) = \log x \) at 1

\( f(x) \) is \( cD \) so this is an over-estimate.

\[ f'(x) = \frac{1}{x} \]

\[ f''(x) = -\frac{1}{x^2} \]

\[ |f''(x)| = \frac{1}{x^2} \leq \frac{1}{(0.9)^2} \text{ for } x \text{ in } [0.9, 1] \]

\[ |\log(0.9) - (-0.1)| \leq \frac{1}{2} \cdot (0.9-1)^2 \]

\[ \frac{10^2}{2 \cdot 9^2} \cdot \frac{1}{10^2} = \frac{1}{2 \cdot 9^2} = \frac{1}{162} \]
Final exam: Same for \( \cos(0.25) \).

**Sol:** \( f(x) = \cos x \)

The tangent of \( f(x) \) at 0 is \( l(x) = 1 \).

\( \cos(0.25) = l(0.25) = 1 \).

Also, \( f''(x) \) is constant, so this is an overestimate.

(also because \(-1 \leq \cos x \leq 1 \)).

\( f'(x) = -\sin x \)

\( \left| f''(x) \right| = |\cos x| = |\cos x| \leq 1 \) for any \( x \).

\( \left| \cos(0.25) - 1 \right| \leq \frac{1}{2} (0.25)^2 = \frac{1}{2} \cdot \frac{1}{4} = \frac{1}{32} \).