Last Week:
- Continuity on intervals + IVT
- First steps with derivatives. - HW 1

Reminder: \( f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \)

* Tangent: \( L(x) = f'(a) \cdot (x - a) + f(a) \)

\( L(x) = f(x) \) for \( x \) near \( a \).

* \( f'(a) \) exists \( \Rightarrow \) \( f(x) \) is cont. at \( a \).

Rules of diff.:
1. \( \frac{dC}{dx} = 0 \) (and if \( \frac{dC}{dx} \to 0 \) for any \( x \) then \( f(x) \) is constant)

2. \( \frac{d(x^n)}{dx} = n \cdot x^{n-1} \) (power rule)

3. \( \frac{d(f \pm g)}{dx} = \frac{df}{dx} \pm \frac{dg}{dx} \) ((\( f \pm g \))') = \( f' \pm g' \)

4. \( \frac{d(cf)}{dx} = c \cdot \frac{df}{dx} \) (c \( f \))' = \( cf' \) (constant multiple)

5. Product Rule: \( \frac{d}{dx}(f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx} \)

6. Power Rule: \( \frac{d}{dx}(f^n) = n \cdot f^{n-1} \cdot \frac{df}{dx} \) \( (e^x)' = e^x \)
[Handwritten text and diagram content]

\[ d(t) = e^t \cdot f \cdot \theta \cdot \phi = \theta(t) \]

Diagram with annotations.
Today:

* More examples
* The quotient rule
* Trig functions.

Next week: Midterm on Monday, June 4th.

Back to the product rule:

\[(f \cdot g)'(a) = \lim_{x \to a} \frac{f(x)g(x) - f(a)g(a)}{x - a}\]

\[
= \lim_{x \to a} \frac{f(x)g(x) - f(a)g(x) + f(a)g(x) - f(a)g(a)}{x - a}
= \lim_{x \to a} \frac{f(x)(g(x) - g(a))}{x - a} + \lim_{x \to a} \frac{f(a)(g(x) - g(a))}{x - a}

= \lim_{x \to a} \frac{f(x) - f(a)g(x)}{x - a} + \lim_{x \to a} \frac{f(a)(g(x) - g(a))}{x - a}.

= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \cdot \lim_{x \to a} g(x) + \lim_{x \to a} f(a) \cdot \frac{g(x) - g(a)}{x - a}.

= f'(a)g(a) + f(a)g'(a).
The Quotient Rule:

\[
\frac{d}{dx} \left( \frac{f}{g} \right) = \frac{g \cdot f' - f \cdot g'}{g^2}
\]

**Example:**

1. \( h(x) = \frac{x+1}{3-x} \)
   
   \( f' = 1 \)
   
   \( g' = -1 \)
   
   \[
   h'(x) = \frac{f \cdot g' - f' \cdot g}{g^2} = \frac{1 \cdot (3-x) - (x+1) \cdot (-1)}{(3-x)^2} = \frac{4}{(3-x)^2}
   \]

2. \( h'(x) = \) where \( h(x) = \frac{e^x + x}{x^2 - 1} \)
   
   \( f' = e^x + 1 \)
   
   \( g' = 2x \)
   
   \[
   h'(x) = \frac{f' \cdot g - f \cdot g'}{g^2} = \frac{(e^x + 1) \cdot (x^2 - 1) - (e^x + x) \cdot 2x}{(x^2 - 1)^2}
   \]
\[
\frac{d}{dx} (x^n) = ? \quad \text{when } n < 0 \text{ integer.}
\]
\[
x^n = \frac{1}{x^{-n}}
\]
\[
f = 1 \quad g = x^{-n} \quad (-n > 0) \quad \text{integer}
\]
\[
f' = 0 \quad g' = (-n)x^{-n-1}
\]
\[
\frac{d}{dx} (x^n) = \frac{d}{dx} \left( \frac{1}{x^{-n}} \right) = \frac{0 \cdot x^{-n} - 1 \cdot (-n)x^{-n-1}}{x^{-2n}}
\]
\[
= n \cdot x^{-n-1}
\]
\[
= n \cdot x^{-n-1} \quad \frac{x}{x^{-2n}} = n x^{n-1}
\]

In fact, for any real \( x \):
\[
\frac{d}{dx} (x^\alpha) = \alpha x^{\alpha-1}
\]

Example:
\[
\frac{d}{dx} (\sqrt{x}) = \frac{d}{dx} (x^{\frac{1}{2}}) = \frac{1}{2} x^{\frac{1}{2} - 1} = \frac{1}{2} x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}
\]
A positive angle $\theta$ results from a counterclockwise rotation.
Figure 1.64

\[
\frac{\varepsilon}{\nu g} = \theta
\]
Figure 1.65
Figure 1.63

Diagram showing the unit circle with angles in degrees and radians, including key points and their coordinates.

Key Points:
- $0^\circ = 0$ radians
- $90^\circ = \pi/2$
- $180^\circ = \pi$
- $270^\circ = 3\pi/2$
- $360^\circ = 2\pi$

Angles and Coordinates:
- $30^\circ = \pi/6$
- $45^\circ = \pi/4$
- $60^\circ = \pi/3$
- $120^\circ = 2\pi/3$
- $135^\circ = 3\pi/4$
- $150^\circ = 5\pi/6$
- $210^\circ = 7\pi/6$
- $225^\circ = 5\pi/4$
- $240^\circ = 4\pi/3$
- $270^\circ = 3\pi/2$

Coordinates:
- $(1, 0)$
- $(0, 1)$
- $(-1, 0)$
- $(0, -1)$
- $(\sqrt{3}, 0)$
- $(0, \sqrt{3})$
- $(-\sqrt{3}, 0)$
- $(0, -\sqrt{3})$
The graphs of $\theta$, $\cos \theta$, and $\sec \theta$ are shown in the left panel. The graphs of $\theta$, $\sin \theta$, and $\csc \theta$ are shown in the right panel. The graphs of $\theta$, $\cos \theta$, and $\sec \theta$ are reciprocal, and the graphs of $\theta$, $\sin \theta$, and $\csc \theta$ are reciprocal.

Figure 1.66 (a) & (b)
The graph of \( y = \tan \theta \) has period \( \pi \).

The graph of \( y = \cot \theta \) has period \( \pi \).

Figure 1.67 (a) & (b)
Derivatives of trig functions:

1. \( \frac{d}{dx} (\sin x) = \cos x \)

2. \( \frac{d}{dx} (\cos x) = -\sin x \)

3. \( \frac{d}{dx} (\tan x) = \frac{d}{dx} \left( \frac{\sin x}{\cos x} \right) = \frac{\cos x \cdot \cos x - \sin x \cdot (-\sin x)}{\cos^2 x} = \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x} \)

\[ \cos^2 x + \sin^2 x = 1 \]

4. \( \frac{d}{dx} (\cot x) = \frac{d}{dx} \left( \frac{\cos x}{\sin x} \right) = -\frac{1}{\sin^2 x} \)
Yesterday:
- Quotient rule
- Trig functions

Today:
- High order derivatives
- Chain rule
- Derivatives of roots, logs, inverse trig and inverse functions in general
- HW1 return in office hours.

High order derivatives:

If \( f(x) \) is differentiable on \((a, b)\)

\[ f'(x) \] is a function on \((a, b)\)

If \( f'(x) \) is differentiable on \((a, b)\) its derivative \( f''(x) \) is a function on \((a, b)\)

\[ f''(x) \] 2nd der.

\[ f'''(x) \] 3rd der.

\[ f''''(x) \] 4th der.

\[ f'''''(x) \] 5th der.
n-th derivative:
\[ f^{(n)}(x) = \frac{d^n f}{dx^n} = \left[ f^{(n-1)} \right]'(x) \]

Examples:
1. \( f(x) = \sin x \)
   \[ f'(x) = \cos x \]
   \[ f''(x) = f'''(x) = (\cos x)' = -\sin x \]
   \[ f^{(4)}(x) = (\cos x)' = \sin x \]

2. \( f(x) = \cos x \)
   \[ f'(x) = \cos x \]
   \[ f''(x) = \cos x \]

3. \( f(x) = e^x \)
   \[ f^{(8)}(x) = e^x \]

4. \( f(x) = Ax^2 + Bx + C \)
   \[ f'(x) = 2Ax + B \]
   \[ f^{(3)}(x) = 0 \]
The Chain Rule:
We want to be able to differentiate the composition of two functions.

Example:
\[ e^{x^2+1} = f(g(x)); \quad g(x) = x^2+1 \]
\[ g(f(x)) = g(e^x) = (e^x)^2 + 1 = e^{2x} + 1 \]

Example: For every $\$10$ discount, a store sells 6 TVs more each week and the price of a TV is going down in the rate of $8.5$ a week.

What is the rate in which the rate of TVs sold increases?

Assume linear demand.

Solution: The rate is 3 TVs/week.

\[
\begin{array}{c|c|c}
\text{P - Price (\$)} & \frac{\Delta q}{\Delta P} = \frac{6}{-10} = -\frac{3}{5} & \frac{\Delta q}{\Delta t} = \frac{\Delta q}{\Delta P} \frac{\Delta P}{\Delta t} \\
\text{q - demand (TVs)} & \frac{\Delta P}{\Delta t} = -\frac{5}{1} = -5 & = -\frac{3}{5} \\
\text{t - time (weeks)} & & \frac{\text{TVs/week}}{5} = 3
\end{array}
\]
Theorem 3.14 (The Chain Rule):

Suppose $y = f(u)$ is diff. at $u = g(x)$ and $u = g(x)$ is diff. at $x$.

Then the composite function $y = f(g(x))$ is diff. at $x$ and its derivative can be expressed in the following ways:

$$\left( f(g(x)) \right)' = f'(g(x)) \cdot g'(x)$$

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

How to use:

1. Identify $f$ and $g$.
2. Write $u = g(x)$, $y = f(u)$
3. Calculate $\frac{dy}{du} = f'(u)$, $\frac{du}{dx} = g'(x)$
4. Substitute $\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$
Examples:

1. \( \frac{d}{dx} \left( e^{x^2+1} \right) = ? \)

\[ y = f(u) = e^u \quad u = g(x) = x^2 + 1 \]

\[ f'(u) = e^u \quad \quad \frac{dy}{du} = e^u \]

\[ g'(x) = 2x \quad \quad \frac{du}{dx} = 2x \]

\[ \frac{d}{dx} (e^{x^2+1}) = f'(g(x)) \cdot g'(x) \]

\[ = f'(x^2+1) \cdot (2x) \]

\[ = e^{x^2+1} \cdot (2x) \]

2. \( \frac{d}{dx} \left( \sin^2 x \right) = ? \)

\[ (f(g(x)))' = f'(g(x)) \cdot g'(x) \]

\[ f'(u) = 2u \quad f'(u) = 2u \]

\[ g'(x) = \sin x \quad g'(x) = \cos x \]

\[ = f'(\sin x) \cdot \cos x \]

\[ = (2 \cdot \sin x) \cdot \cos x \]
\[ \frac{d}{dx} \left( \sin^2 x \right) = \frac{d}{dx} \left( \sin x \cdot \sin x \right) \]

\[ = \cos x \cdot \sin x + \sin x \cdot \cos x = 2 \sin x \cdot \cos x. \]

\[ \frac{d}{dx} \left( \sin^2 x \right) = \frac{d}{dx} \left( \frac{1 - \cos(2x)}{2} \right) \]

\[ = \frac{d}{dx} \left( \frac{1}{2} - \frac{1}{2} \cos(2x) \right) = -\frac{1}{2} \frac{d}{dx} \left( \cos(2x) \right) \]

\[ = -\frac{1}{2} \left[ -\sin(2x) \cdot 2 \right] = \sin(2x) \]

\[ = 2 \sin x \cdot \cos x \]

\[ (3) \quad \frac{d}{dx} \left( f(x)^n \right) = n \left( f(x) \right)^{n-1} \cdot f'(x) \]

(The Power Rule)
4) \[ \frac{d}{dx} \left( \sin \left( e^{3 \cos x} \right) \right) \]

\[ = \cos \left( e^{3 \cos x} \right) \cdot \left[ e^{3 \cos x} \right]' \]

\[ = \cos \left( e^{3 \cos x} \right) e^{3 \cos x} \cdot \left[ 3 \cos x \right]' \]

\[ = -3 \cos \left( e^{3 \cos x} \right) e^{3 \cos x} \cdot \sin x \]

5) \[ \frac{d}{dx} \left( \log x \right) = ? \]

\[ e^{\log x} = x \]

\[ e^{\log x} \cdot \left( \log x \right)' = 1 \]

\[ x \cdot \left( \log x \right)' = 1 \]

\[ \frac{d}{dx} \left( \log x \right) = \frac{1}{x} \]
\( \frac{d}{dx} \left( \frac{\log(x^2 + \sin(e^x))}{u} \right) \)

\[ \frac{d}{du} \left( \frac{1}{u} \right) \cdot \frac{du}{dx} = \frac{1}{x^2 + \sin(e^x)} \cdot (x^2 + \sin(e^x)) \]

\[ = \frac{1}{x^2 + \sin(e^x)} \left[ 2x + \cos(e^x) \cdot e^x \right] \]

\( \frac{d}{dx} \left( x^x \right) = ? \)

\[ x^x = \left( x \right)^x = \left( e^{\log x} \right)^x = e^{x \log x} \]

\[ \frac{d}{dx} (x^x) = \frac{d}{dx} (e^{x \log x}) = e^{x \log x} \left( x \frac{d}{dx} \log x \right) \]

\[ = e^{x \log x} \cdot \left( \frac{1}{x} \log x + x \cdot \frac{1}{x} \right) = x^x \left( \log x + 1 \right) \]

\( \frac{d}{dx} (e^{f(x)}) = e^{f(x)} \cdot f'(x) \)

\( \frac{d}{dx} (\ln \log x) = \frac{1}{\ln x} \)
Derivatives of Inverse functions:

Examples:

1. \( \frac{d}{dx}(\log x) = \frac{1}{x} \)

2. \( f(x) = \sqrt{x} \) the inverse of \( x^2 \),
   \[(\sqrt{x})^2 = x \quad (x > 0)\]

0. If:
   \[2 \cdot (\sqrt{x}) \cdot \left[\sqrt{x}\right]' = 1\]

\[\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}\]

2. \( \frac{d}{dx}(\sqrt[n]{x}) = ? \)

\[\left(\sqrt[n]{x}\right)^n = x \quad x > 0 \quad \text{if } n \text{ is even}\]

\[n \left(\frac{x}{n}\right)^{n-1} \cdot \left(\sqrt[n]{x}\right)' = 1\]

\[\frac{d}{dx} \left( x^{\frac{1}{n}} \right) = \frac{d}{dx} (\sqrt[n]{x}) = \frac{1}{n} x^{\frac{1}{n} - 1}\]
Inverse Trig. Functions:

\[ \frac{d}{dx}(\arcsin x) = ? \]

\[ \sin(\arcsin x) = x \]

\[ \cos(\arcsin x) \cdot [\arcsin x]' = 1 \]

\[ \frac{d}{dx}[\arcsin x] = \frac{1}{\cos(\arcsin x)} \approx \frac{1}{\sqrt{1-x^2}} \]

\[ x = \sin(y) \]

\[ \frac{d}{dx}(\arccos x) = \frac{-1}{\sqrt{1-x^2}} \]

\[ \frac{d}{dx}(\arctan x) = \frac{1}{1+x^2} \]
\[ \{ x \in \mathbb{R} : 0 < x < \frac{\pi}{2} \} \text{ is the value of } x \text{ such that } \csc x = \frac{1}{x}. \]

\[ \{ x \in \mathbb{R} : x > 0 \} \text{ is the value of } x \text{ such that } \sec x = \frac{1}{x}. \]

\[ \{ x \in \mathbb{R} : x < 0 \} \text{ is the value of } x \text{ such that } \csc x = \frac{1}{x}. \]

\[ \{ x \in \mathbb{R} : x < 0 \} \text{ is the value of } x \text{ such that } \sec x = \frac{1}{x}. \]

\[ \{ x \in \mathbb{R} : x > 0 \} \text{ is the value of } x \text{ such that } \cot x = \frac{1}{x}. \]

\[ \{ x \in \mathbb{R} : x > 0 \} \text{ is the value of } x \text{ such that } \tan x = \frac{1}{x}. \]
Theorem: (Derivatives of Inverse Functions)
Assume that near \( x = a \), \( f(x) \) is invertible,
\( b = f(a) \) and assume that \( f(x) \) is
diff. at \( a \) with \( f'(a) \neq 0 \).
Then \( f^{-1} \) is diff. at \( b \) and
\[
(f^{-1})'(b) = \frac{1}{f'(a)}
\]

\[
f(f^{-1}(x)) = x
\]

iff:
\[
f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1
\]

\[
(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}
\]
Yesterday:
- High order derivatives
- Chain Rule
- Derivative of inverse functions.

Today:
- Implicit differentiation
- Logarithmic differentiation
- Normal lines.

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Warm Up Problems:

1. Find \( \frac{d}{dx} (\sin x \cos x) \) (For \( x \) s.t. \( \sin x > 0 \))

\[
\frac{d}{dx} (\sin x \cos x) = \frac{d}{dx} \left[ \left( \log (\sin x) \right) \cos x \right] = \frac{d}{dx} \left[ \cos x \cdot \log (\sin x) \right]
\]

\[
= \cos x \cdot \log (\sin x) - \sin x \cdot \cos x \cdot \frac{1}{\sin x} \cdot \cos x
\]

2. Let \( y = 3x + 1 \) be the tangent of \( f(x) \) at \( x = 2 \) and let \( y = 4x - 2 \) be the tangent of \( g(x) \) at \( x = 1 \).

@ Find the line equation for the tangent of \( f(g(x)) \) at \( x = 1 \).

@ Find the line equation of \( f^{-1}(x) \) at \( f(2) \).

Solution @ \( f(1) = 4 \cdot 1 - 2 = 2 \)

\( f(2) = 3 \cdot 2 + 1 = 7 \).

\( f(g(1)) = f(2) = 7 \)

(1, 7)
The slope of the tangent at $x=1$ will be:

$$f'(y)(1) = f'(2) \cdot g'(1)$$

$$= 3 \cdot 4 = 12$$

$$y = 12(x-1) + 7 = 12x - 5$$

Alternative Solution:
The tangent of $f(u)$ at $u=2$ is $y = 3u+1$.
The tangent of $g(x)$ at $x=1$ is $u = 4x-2$.

$$y = 3u+1 = 3(4x-2) + 1 = 12x - 5$$

(b) $f(2) = 2$

$$f^{-1}(7) = 2$$

$$(f^{-1})'(7) = \frac{1}{f'(2)} = \frac{1}{3}$$

$$y = \frac{1}{3}(x-7) + 2 = \frac{1}{3}x - \frac{1}{3}$$

Alternative Solution:
The tangent of $f(x)$ at $x=2$ is $y = 3x+1$.
Solve for $x$:

$$x = \frac{1}{3}(y-1) = \frac{1}{3}y - \frac{1}{3}$$

So:

$$u = \frac{1}{3}y - \frac{1}{3}$$
Logarithmic differentiation:

Note that:

* \( \frac{d}{dx} (\log(x)) = \frac{1}{x} \quad x > 0 \)

* \( \frac{d}{dx} (\log(1+x)) = \frac{1}{x} \quad x 
eq 0 \)

Why? We need to explain only for \( x < 0 \):

\[
\frac{d}{dx} (\log(x)) = \frac{d}{dx} (\log(-x)) = \frac{1}{-x} \cdot (-x)^{-1} = \frac{1}{x}
\]

* \( \frac{d}{dx} (\log(f(x))) = \frac{f'(x)}{f(x)} \quad (f(x) \neq 0) \)

Examples:

1. Find \( \frac{d}{dx} \), where \( y = \frac{x^5}{(1-10x)\sqrt{x^2+2}} \)

\[
\log(y) = \log\left(\frac{x^5}{(1-10x)\sqrt{x^2+2}}\right) = \log(x^5) - \log((1-10x)\sqrt{x^2+2})
\]

\[
= \log(x^5) - \log(1-10x) - \log(x^2+2)
\]

\[
= 5 \log x - \log(1-10x) - \frac{1}{2} \log(x^2+2)
\]
Lift:
\[ y' = \frac{5}{x} + \frac{10}{1-10x} - \frac{1}{2} \frac{2x}{x^2+2} \]

\[ \Rightarrow y' = y \cdot \left( \frac{5}{x} + \frac{10}{1-10x} - \frac{x}{x^2+2} \right) \]

(2) \[ y' = ? \] Where \[ y = \sin x \cos x \]

\[ \log(y) = \log(\sin x \cos x) = \cos x \cdot \log(\sin x) \]

\[ \Rightarrow \frac{y'}{y} = -\sin x \cdot \frac{\log(\sin x)}{\sin x} + \cos x \cdot \frac{1}{\sin x} \cdot \cos x \]

\[ y' = y \cdot \left( \frac{\cos^2 x}{\sin x} - \sin x \cdot \log(\sin x) \right) \]

Note that:
\[ \frac{d}{dx} (\log_b(x)) = \frac{d}{dx} \left( \frac{\log(x)}{\log(b)} \right) = \frac{1}{\log(b) \cdot x} \]
Implicit Differentiation:

Consider the circle \( x^2 + y^2 = 1 \)

we want a line eq. for the tangent at \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \)

Let's diff. \( x^2 + y^2 = 1 \) (w.r.t. \( x \))

\[
2x + 2y \cdot y' = 0
\]

\[
\frac{d}{dx}(y^2) = \frac{d(y^2)}{dy} \cdot \frac{dy}{dx}
\]

\[
\frac{1}{2y} \cdot \frac{dy}{dx}
\]

Solving for \( y' \) we find:

\[
\begin{array}{c}
\text{at } \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \\
y' = -\frac{x}{y}
\end{array}
\]

\[ y' \left( \frac{1}{\sqrt{2}} \right) = -\frac{\frac{1}{\sqrt{2}}}{\frac{1}{\sqrt{2}}} = -\frac{\sqrt{2}}{\sqrt{2}} = -1
\]

So the tangent to \( x^2 + y^2 = 1 \) at \( \left( \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \) is:

\[
y = (-1) \cdot (x - \frac{1}{\sqrt{2}}) + \frac{1}{\sqrt{2}} = -x + \frac{2}{\sqrt{2}}
\]
Figure 3.52 (a & b)

$x^2 + y^2 = 1$ fails the vertical line test.

Each semicircle passes the vertical line test.
Ex: Given the demand equation for a product:

\[ q^2 + p^{3/2} + 2p = 20 \]

Find \( \frac{dq}{dp} \) at \( p = \$4 \).

Solution:

Let's diff. \[ q^2 + p^{3/2} + 2p = 20 \]

\[ 2q \cdot \frac{dq}{dp} + \frac{3}{2} p^{1/2} + 2 = 0 \]

\[ \frac{dq}{dp} = \frac{-2 - \frac{3}{2} p^{1/2}}{2q} \]

\[ q(\$4) = \frac{-2 - \frac{3}{2} \cdot 4^{1/2}}{2 \cdot q(\$4)} \]

\[ q(\$4) = -2 - \frac{3}{2} \cdot 2 = -5/4 \]

The tangent of \( q(p) \) at \((4, 2)\) is:

\[ \theta(p) = -\frac{5}{4}(p - 4) + 2 \]
Example: \( \frac{d}{dx} \left( x^{\frac{n}{m}} \right) = ? \)

\[ y = x^{\frac{n}{m}} \]

\[ y^m = x^n \]

\[ y' = \frac{n}{m} \cdot x^{\frac{n}{m} - 1} \]

\[ y' = \frac{n}{m} \cdot \frac{x^{n-1}}{x^{\frac{n}{m}(m-1)}} = \frac{n}{m} \cdot x^{n-1} \cdot x^{\frac{-n}{m}} \]

Example: Find \( y'(x) \) where \( y(x) \) satisfy

\[ x + y^2 - xy = 1 \]

Solution:

\[ \text{diff. w.r.t. } x: \quad 1 + 3y^2y' - (y + xy') = 0 \]

Solve for \( y' \):

\[ 3y^2y' - xy' = y - 1 \]

\[ (3y^2 - x)y' = y - 1 \]

\[ y'(x) = \frac{y - 1}{3y^2 - x} \]
Figure 3.53 (a)

Graph of $x + y^3 - xy = 1...$
Final 2012:
Consider the curve \( x^2 + y^3 - 2xy = 0 \).
Find \( f''(1) \) assuming that \( y = f(x) \) near \( (x,y) = (1,1) \).

Solution:

\[
\text{Diff. } x^2 + y^3 - 2xy = 0 \quad \text{w.r.t. } x:
\]

\[
2x + 3y^2 y' - 2y - 2xy' = 0
\]

\[
\frac{d}{dx}(-2xy) = -2 \frac{d}{dx}(xy) = -2[y + xy']
\]

\[
2 + 3 \left[ (2y y') y' + y^2 \cdot y'' \right] - 2y' - 2[1 \cdot y' + xy'] = 0
\]

Let's find \( f'(1) \): plug \( x = 1 \), \( y = 1 \) into \( \bigcirc \):

\[
2 \cdot 1 + 3 \cdot 1^2 \cdot f'(1) - 2 \cdot 1 - 2 \cdot 1 \cdot f'(1) = 0
\]

Solving for \( f'(1) \) we get \( f'(1) = 0 \).

Plugging \( x = 1 \), \( y = 1 \), \( y' = 0 \) into \( \bigcirc \) we get:

\[
2 + 3 \left[ 2 \cdot 1 \cdot 0 + 12 \cdot f''(1) \right] - 2 \cdot 0 - 2 \cdot [1 \cdot 0 + 1 \cdot f''(1)] = 0
\]

\[
\frac{f''(1)}{f''(1)} = -2
\]
Normal line to a Curve:

* A normal line to \( y = f(x) \) at \( x = a \) is the line perpendicular to the tangent at \( y = f(x) \) at \( x = a \). \((a, f(a))\)

* The slope of the tangent at \( x = a \) is \( f'(a) \neq 0 \)

So the slope of the normal line at \( x = a \) would be \( \frac{-1}{f'(a)} \)

Examples: \( x^2 + y^2 = 1 \)

Find the normal line to the circle at \( (\frac{1}{2}, \frac{\sqrt{3}}{2}) \)

Solution: We know that \( y = -\frac{x}{y} \)

So the slope of the tangent at this point is \( y'(\frac{1}{2}) = -\frac{\frac{\sqrt{3}}{2}}{\frac{\sqrt{3}}{2}} = -\frac{1}{\sqrt{3}} \)
The slope of the normal line is

\[- \frac{1}{y'(\frac{1}{2})} = -\frac{1}{\frac{\sqrt{3}}{2}} = \frac{2}{\sqrt{3}} = \sqrt{3}\]

The normal line equation is:

\[y = \sqrt{3} \cdot (x - \frac{1}{2}) + \frac{\sqrt{3}}{2} = \frac{\sqrt{3}}{2}x\]
Figure 3.54

$x^2 + y^2 = 1$ has two tangent lines at $x = \frac{1}{2}$. 

\[ \text{Slope} = -\frac{1}{\sqrt{3}} \]

\[ \text{Slope} = \frac{1}{\sqrt{3}} \]
Example: Find the normal line to the curve given by $(x+y)^3 = x^3 + y^3$ at $(-1, 1)$.

Solution:

\[ \text{Diff.} \quad (x+y)^3 = x^3 + y^3 \quad \text{w.r.t.} \quad x \]

\[ 3(x+y)^2 \cdot (1+y') = 3x^2 + 3y^2 \cdot y' \]

Plug in $x = -1$, $y = 1$:

\[ 0 = 3 \cdot (-1 + 1) \cdot (1 + y'(-1)) = 3(-1)^2 + 3 \cdot \frac{2 \cdot y'(-1)}{3} \]

\[ y'(-1) = -1 \]

So, the slope of the normal is $-\frac{1}{y'(-1)} = 1$

So, the normal line is \[ y = 1 \cdot (x - (-1)) + 1 = x + 2 \]
Yesterday:
- Logarithmic diff.
- Implicit diff.
- Normal lines

Today:
- The derivative as a rate of change.
- Related Rates
- Revisions.

Derivatives as Rates of Change:

$f'(x)$ describes the rate in which $f(x)$ changes near $x$.

In particular, if $h(t)$ is the height of an object at time $t$:

$s(t) = h'(t)$ - instantaneous velocity.

$s(t) = |h(t)|$ - instantaneous speed.

$a(t) = s'(t) = h''(t)$ - inst. acceleration.

Example: How fast is the stone hitting the ground when $h(t) = -5t^2 + 30t$? (ground is at height 0)
Solution: \( u(t) = h'(t) = -10t + 30 \)

Let's solve \( h(t) = 0 \):

\[-5t^2 + 30t = 0\]

\[t(-5t + 30) = 0\]

\[t = 0 \quad \text{or} \quad -5t + 30 = 0\]

\[t = 6 \text{ seconds}\]

\[u(6) = -10 \cdot 6 + 30 = -30 \text{ m/s}\]

\[S(6) = 30 \text{ m/s}\]

Example: What is the maximal height that the stone reaches?

Solution: Show: \( u(t) = 0 \)

\[-10t + 30 = 0\]

\[t = 3 \text{ seconds}\]

When the stone is at max. height:

\[h(3) = -5 \cdot 3^2 + 30 \cdot 3 = 45 \text{ meters}\]
Steps for Rate-Related Problems

1. Read the problem carefully, making a sketch to organize the given information.
2. Identify the rates that are given and the rate that is to be determined.
3. Introduce rates of change by differentiating the appropriate equation(s) with variables.
4. Substitute known values and solve for the desired quantity.
5. Check that units are consistent and the answer is reasonable. (For example, does it have the correct sign?)
Related Rates:

Q1: Suppose that air is being pumped into a spherical balloon at a rate of 100 cm³/sec. How fast is the diameter increasing when it is 50 cm?

Solution:

\[ V = \frac{4}{3} \pi R^3 = \frac{4}{3} \left( \frac{D}{2} \right)^3 = \frac{\pi}{6} D^3 \]

\[ \frac{dV}{dt} = 100 \text{ cm}^3/\text{sec} \]

\[ \frac{dV}{dt} = \frac{d}{dt} \left( \frac{\pi}{6} D^3 \right) = \frac{d}{dD} \left( \frac{\pi}{6} D^3 \right) \cdot \frac{dD}{dt} = \frac{\pi}{2} D^2 \frac{dD}{dt} \]

Plug \( \frac{dV}{dt} = 100 \), \( D = 50 \) to get:

\[ 100 = \frac{\pi}{2} \cdot 50^2 \cdot \frac{dD}{dt} \]

\[ \frac{dD}{dt} = \frac{2 \cdot 200}{\frac{\pi}{2} \cdot 50^2} = \frac{2}{50 \pi} \text{ cm/see} \]
\[
\frac{d}{dt} \left( \frac{1}{2} D^3 \right) = \frac{1}{6} \left( \frac{1}{3} \sqrt{D(t)^3} \right)^3
\]

\[= \frac{\pi}{6} \cdot 3 \cdot D(t)^2 \cdot D'(t) \]

241: One end of a 4 meters ladder is on the ground and the other end rests on a vertical wall. If the bottom end slides from the wall at the rate of 1 m/sec, how fast is the top of the ladder sliding down the wall when the bottom of the ladder is 2 meters from the wall?

Solution: A - distance of bot. from wall \[\text{[m]}\]
B - \(-11\) - top -11- floor \[\text{[m]}\]
\(t\) - time \[\text{[sec]}\]

\(-\frac{d}{dt}B\mid_{AB=2} = ?\)
\[ A^2 + B^2 = 4^2 = 16 \]

\[ \frac{d}{dt} : 2A \frac{dA}{dt} + 2B \frac{dB}{dt} = 0 \]

When \( A = 2 \), we have \( 2^2 + B^2 = 16 \)
\[ \Rightarrow B^2 = 12 \quad \Rightarrow \quad B = \pm \sqrt{12} \quad [m] \]

Plug \( A = 2, \frac{dA}{dt} = 1, B = \sqrt{12} \)

\[ 2 \cdot 1 + \sqrt{12} \cdot \frac{dB}{dt} \bigg|_{A=2} = 0 \]

\[ \frac{dB}{dt} \bigg|_{A=2} = -\frac{2}{\sqrt{12}} \quad \text{m/sec} \]

Final answer: \( \frac{1}{\sqrt{12}} \text{ m/sec} \)
Q9: General Foods Cereals TM makes 5 thousand pecks of Fruit-loop each week when the wholesale price is $p per box. The demand equation is \( 6q^2 - 5pq + 2p^3 = 5 \). How fast is the supply of cereals changing when the price per box is $6.50 and the quantity is 10,000 boxes, and the wholesale price per box is increasing at the rate of 10c per box each week?

Solution: 

\[ p \text{ - price } [\$] \]
\[ q \text{ - demand } [\text{kilo box} - 1000 \text{ boxes}] \]
\[ t \text{ - time } [\text{weeks}] \]

\[ 6q^2 - 5pq + 2p^3 = 5 \]

\[ \frac{dq}{dt} \bigg|_{p=6.50, q=10} = ? \]
\[ \frac{d^2}{dt^2} - 5 \left[ \frac{d}{dt} \cdot p + q \cdot \frac{dp}{dt} \right] + 2.3 p^2 \frac{dp}{dt} = 0 \]

Plug in: \( p = 6.5 \), \( q = 10 \), \( \frac{dp}{dt} = 0.1 \)

\[ 12 \cdot 10 \cdot \frac{dp}{dt} - 5 \left[ \frac{d}{dt} \cdot 6.5 + 10 \cdot 0.1 \right] + 2.3 (6.5)^2 \cdot 0.1 = 0 \]

\[ \frac{dp}{dt} = \ldots \]

\[ p = 6.5 \]
\[ q = 10 \]

Q6: While in Wonderland, Alice eats a cookie that makes her grow taller at a rate of \( \frac{1}{2} \) m/sec. If she is standing 20 m from a light which is 10 m tall, how fast is the length of her shadow changing when she is 5 m tall?
60% General Foods Cereal, Inc. makes 9 thousand packs of Fruit-loops cereal each week. When the wholesale price per box is $6.50, the quantity is 10,000 boxes (q = 10) and the wholesale price per box is increasing at the rate of 0.05 per box each week. Suppose that air is being pumped into a spherical balloon at a rate of 100 cm$^3$/sec. How fast is the diameter changing when the radius is 20 cm? Rate of 2 m$^2$/min. How fast is the radius of the slick increasing when its radius is 10 meters? Decreasing?
\[
\frac{H}{L} = \frac{10}{20 + L}
\]

\[H \cdot (20 + L) = 10 \cdot L\]

\[
\frac{dL}{dt} = \frac{dH}{dt} (20 + L) + H \cdot \frac{dL}{dt} = 10 \cdot \frac{dL}{dt}
\]

Plug in: \(\frac{dH}{dt} = \frac{1}{2}, H = 5, L = 20\)

\[
\frac{1}{2} (20 + 20) + 5 \cdot \frac{dL}{dt} = 10 \cdot \frac{dL}{dt}
\]

\[
20 = 5 \frac{dL}{dt}
\]

\[
\frac{dL}{dt} = 4 \text{ m/sec}
\]
HW1 Q4:

Little Johnny:

\[ p = 81 \rightarrow q = 100 \text{ ups/week} \]
\[ \Delta p = 0.1 \rightarrow \Delta q = -20 \]

\[ p = Aq + B \]
\[ A = \text{slope} = \frac{\Delta p}{\Delta q} = \frac{0.1}{-20} = -\frac{1}{200} \]

Plug in \( p = 1, q = 100 \) \( A = -\frac{1}{200} \)

\[ 1 = -\frac{1}{200} \cdot 100 + B \rightarrow B = 3.5 \]

\[ p = -\frac{1}{200} p + 3.5 \]

\[ \mathcal{R} = pq \]
\[ = q \left( -\frac{1}{200} q + 3.5 \right) \]
\[ R = q \left( -\frac{1}{200} q + \frac{3}{2} \right) \]

\[ R = p \cdot (-200p + 300) = 0 \]

\[ p = 0 \quad \text{or} \quad p = \frac{3}{2} \]

max is at \( p = \frac{3}{4} \).

\[ f(x) = Ax^2 + Bx + C \quad - \frac{B}{2A} \]

\[ f'(x) = 2Ax + B = 0 \]

\[ x = -\frac{B}{2A} \]
Example: \( y = x^2 + 2 \)

It is one to one on \((0, \infty)\)

\(\Rightarrow\) it has an inverse, \( f^{-1}(x) \)

\[ y = f^{-1}(x) = \sqrt{x - 2} \quad x > 2 \]

Write \( y = f^{-1}(x) \)

we know that \( x = y^2 + 2 \)

\[ \frac{d}{dx} : 1 = 2y \cdot y' \]

\(\Rightarrow\) \[ y' = \frac{1}{2y} = \frac{1}{2\sqrt{x - 2}} \]