Last week:
- Introduction
- Limits
- Continuity

This week:
- More about continuity and what is it good for?
- Derivatives

Warm-Up problem:

Final 2011: Suppose that \( f(x) \) and \( g(x) \) are continuous functions at \( x=3 \) and assume that \( g(3)=2 \) \( \Rightarrow \) and \( \lim_{x \to 3} (x f(x) + g(x)) = 1 \)

Find \( f(3) \).

Find the maximal interval on which \( f(x) \) is defined.
Solution: By the assumption that \( f(x) \) and \( g(x) \) are cont. at \( x=3 \), we have:

\[
\lim_{x \to 3} f(x) = f(3) \quad \text{and} \quad \lim_{x \to 3} g(x) = g(3) = 2.
\]

Also:

\[
1 = \lim_{x \to 3} (xf(x) + g(x)) = \lim_{x \to 3} xf(x) + \lim_{x \to 3} g(x) = 3f(3) + 2
\]

\[
= \left( \lim_{x \to 3} x \right) \left( \lim_{x \to 3} f(x) \right) + 2 = 3f(3) + 2
\]

\[
= 3f(3) + 2 \Rightarrow f(3) = -\frac{1}{3}
\]

Alternatively: \( f(x), g(x), x \) are all cont. at \( x=3 \) \( \Rightarrow \) \( xf(x) + g(x) \) is cont. at 3.

hence:

\[
\lim_{x \to 3} (xf(x) + g(x)) = 3f(3) + g(3).
\]
In order for \( f \) to be continuous at \( a \), the following three conditions must hold:

1. \( (p) f \) is defined at \( a \) in the domain of \( f \).
2. \( \lim_{x \to a} (x) f \) exists.
3. \( \lim_{x \to a} (p) f = (x) f \) exists.

\( (p) f = (x) f \iff \lim_{x \to a} (p) f = (x) f \) in the limit of \( f \) at \( a \).
\[ f(x) \]  
\[ u((x)f) \quad \text{if} \]  
\[ \delta \quad \text{provided } \delta / f \]  
\[ p \quad \text{if} \]  
\[ \delta - f \quad p \]  
\[ 0 \neq (v)(x) \]  
\[ \text{THEOREM 2.9} \]  

Assume \( c \) is a constant and \( n \) is an integer. If \( f \) and \( g \) are continuous at \( a \), then the following functions are also continuous at \( a \).
b. A rational function (a function of the form \( \frac{b}{d} \), where \( b \) and \( d \) are polynomials) is continuous for all \( x \).

a. A polynomial function is continuous for all \( x \).

THEOREM 2.10 Polynomial and Rational Functions
Logarithmic

Exponential

Trigonometric

Inverse Trigonometric

The following functions are continuous at all points of their domains.

**Theorem 2.15** Continuity of Transcendental Functions
Example (Final Sept 2013): undeclared constant.
Find the value of \( a \) for which \( f(x) \) is continuous, where
\[
f(x) = \begin{cases} 
\frac{x^2 + 3x + 2}{x + 1}, & x \neq -1 \\
a, & x = -1 
\end{cases}
\]

Solution:

\( f(-1) \) is defined \( f(-1) = a \).

\( \lim_{x \to -1} f(x) \) exists

\[
\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{x^2 + 3x + 2}{x + 1} = \lim_{x \to -1} \frac{(x+1)(x+2)}{(x+1)} = \lim_{x \to -1} (x+2) = -1 + 2 = 1
\]

\( \lim_{x \to -1} f(x) = f(-1) \) when \( a = 1 \).
Composition of Functions: \[ f \]

\[ \xrightarrow{g} \]

\[ f \circ g \]

\[ f(g(a)) \]

**Theorem 2.11**: If \( g \) is cont. at \( a \) and \( f \) is cont. at \( g(a) \) then \( f \circ g \) is continuous at \( a \).

**Theorem 2.12**: If \( g \) is cont. at \( a \) and \( f \) is cont. at \( g(a) \) then

\[
\lim_{{x \to a}} f(g(x)) = f\left(\lim_{{x \to a}} g(x) \right) = f(g(a))
\]
2. If \( \lim_{x \to a} g(x) = L \) and \( f \) is cont at \( L \), then \( \lim_{x \to a} f(g(x)) = f(L) \)

Examples: Things like

\[ e^{\log(\sin(x^2) + 1 - \frac{\cos x}{e^{x+1}})} \]

is cont. on its domain.

(Note: \( f(x) = x^2 \), \( g(x) = \sin x \) are both cont.

For any \( x \) so \( \sin(x^2) = f(g(x)) \)

is also cont. For any \( x \)
Example (Final 2016): Consider the following

Function: \( f(x) = \begin{cases} 4 - x^2, & x < 1 \\ \log(x), & 1 < x < 3 \\ e^x, & x \geq 3 \end{cases} \)

Evaluate \( \lim_{x \to 1^-} f(f(x)) \).

Solution:

\[
\lim_{x \to 1^-} f(x) = \lim_{x \to 1^-} (4 - x^2) = 4 - 1^2 = 3 \]

\[
\lim_{x \to 1^-} f(f(x)) = \lim_{x \to 1^-} f(4 - x^2)
\]

\[
\begin{align*}
\{z = f(x) \} &= \lim_{z \to 3^+} f(z) \\
&= \lim_{z \to 3^+} e^2 = e^3
\end{align*}
\]
any $x \in (-\infty, 0)$
Continuity on Intervals:

Definition: We say that \( f(x) \) is cont. at an interval \((a,b)\) if \( f(x) \) is cont. at any \( x \) in \((a,b)\).

We want to talk about cont on \([a,b]\).

Suggestion: Define it to be: "\( f \) is cont. on \([a,b]\) if it is cont. at any \( x \) in \([a,b]\)".

Problem: \( f(x) = \sqrt{x} \) domain \( x \geq 0 \)

\[
\lim_{{x \to 0}} f(x) \text{ doesn't exist.}
\]

So, in this definition, \( \sqrt{x} \) is not cont. on \([0,1]\).

So we need a better definition for closed intervals.
A function \( f(x) \) is continuous from the right at \( b \) if

\[
\lim_{x \to b^+} f(x) = f(b)
\]

We say that \( f(x) \) is cont. at \( \mathbb{E}(a,b) \) if it is:

- @ cont. at \((a,b)\)
- @ right cont. at \( a \)
- @ left cont. at \( b \).

HW: Define continuity on any other type of interval.
Figure 2.52

Right-continuous

Left-continuous

$g(x) = \sqrt{9 - x^2}$

Continuous on $[-3, 3]$
As a composition of {\text{cut}}, functions on this

so the domain of \( f(x) \) is

\( (\infty, -1) \cup (1, \infty) \)

and of \( g(x) \) is

\( (0, \infty) \cup (-\infty, \infty) \)

The domain of \( h(x) \) is any \( x \) where

\[ f(x) = \begin{cases} \ln(x) & x > 0 \\ 0 & x = 0 \\ x & x < 0 \end{cases} \]

Solution:

\( x \in \{x \mid f(x) = \ln(x) = 1\} = \{x \mid \ln(x) = 1, x > 0\} = \{e\} \)

Exercise: Find the maximal intervals on

which \( f(x) \) is continuous.
Theorem 2.14: If a function \( f(x) \) is continuous and one-to-one on an interval \( I \) then its inverse \( f^{-1} \) is also continuous.

Final 2010: Find the value of \( a \) for which \( f(x) \) is continuous at \( (0, \infty) \),

where \( f(x) = \begin{cases} \text{2x}^2 - \log(x), & x \geq 1 \\ \text{x}^3 - 4ax, & x < 1 \end{cases} \)

Solution:

* For \( x > 1 \): \( f(x) = 2x^2 - \log(x) \) is continuous.
* For \( 0 < x < 1 \): \( f(x) = x^3 - 4ax \) is continuous.

Let's find \( a \) so that \( f(x) \) is continuous at \( x = 1 \).
\[ f(1) = 2 \cdot 1^2 - \log(1) = 2 \quad \text{exists} \]

\[
\lim_{x \to 1} f(x) \quad \text{exists.}
\]

\[
\lim_{x \to 1^+} (2x^2 - \log(x)) = 2 \cdot 1^2 - \log(1) = 2
\]

\[
\lim_{x \to 1^-} (x^3 - 4ax) = 1^3 - 4 \cdot a \cdot 1 = 1 - 4a
\]

\[ 2 = 1 - 4a \]

\[ a = \frac{-1}{4} \]

\[ f(1) = \lim_{x \to 1} f(x) = 2 \]
Yesterday:
- Continuity: Examples + Continuity on intervals.

Today:
- Intermediate Value Theorem (IVT)
- Intro to derivatives.

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**The Intermediate Value Theorem (IVT)**

**Theorem (2.16):**
Suppose $f$ is cont. on $[a, b]$ and $L$ is (strictly) between $f(a)$ and $f(b)$.

Then there exists at least one $(a < c < b)$ such that $f(c) = L$.

Marked list:
1. $f(x)$ - a function
2. $a$, $b$, and $L$
3. $f$ is cont. on $[a, b]$.
4. $L$ is between $f(a)$ and $f(b)$. 
Example: Show that the equation $x^2 = 2$ has a solution.

Solution:

$\begin{align*}
* & \quad f(x) = x^2 - 2 \\
* & \quad a = 1 \quad b = 2 \quad \text{and} \quad L = 0 \quad (\text{so that} \quad f(c) = 0 \\
& \quad \text{means} \quad c^2 - 2 = 0 \quad \text{i.e.} \quad c^2 = 2)
\end{align*}$

(Alternatively, take $f(x) = x^2$ and $L = 2$)

$\begin{align*}
& \left\{ \\
& \quad f(-1) = -1 \\
& \quad f(0) = -2 \\
& \quad f(1) = -1 \\
& \quad f(2) = 2
\end{align*}$

$\begin{align*}
* & \quad f(x) \text{ is cont. on } [1, 2]. \\
* & \quad f(1) < f(0) \\
* & \quad -1 \leq 0 \leq 2
\end{align*}$

So, by IVT, there exists $1 < c < 2$

$s.t. \quad f(c) = 0$, i.e. $c^2 = 2$. 
In $(a, b)$ there is at least one number $c$ such that $f(c) = L$, where $L$ is between $f(a)$ and $f(b)$. 

**Intermediate Value Theorem**

Figure 2.56
Example (Final 2018): Show that the equation 
\[ 5^x = 10x + 7 \]
has a solution.

Solution:

Warning: \( f(x) = 5^x = 10x + 7 \) or \( f(x) = 5^x - 10x - 7 = 0 \)
are not functions.

* \( f(x) = 5^x - 10x - 7 \)

* Take \( L = 0 \)

\[ f(0) = 1 - 0 - 7 = -6 \]
\[ f(1) = 5^1 - 10 - 7 = -12 \]
\[ f(2) = 5^2 - 20 - 7 = -2 \]
\[ f(3) = 5^3 - 30 - 7 = 125 - 37 > 0 \]
\[ a = 2 \text{ and } b = 3, \]
* \( f \) is cont. on \([2,3]\).

* \( f(2) = -2 < 0 < f(3) = 88 \)

So, by IVT, there exists \( 2 < c < 3 \) st. \( f(c) = 0 \)

i.e. \( c \approx 1.39 \).
Example:
Show that the equation \( \log(x) = x - 3 \) has a solution.

Solution:
* \( f(x) = x - \log(x) \) Domain of \( f \) is \( (0, \infty) \)

* \( L = 3 \), \( a = e^1 \), \( b = e^3 \)

\[
\begin{align*}
  f(1) &= 1 - \log(1) = 1 - 0 = 1 < 3 \\
  f(e) &= e - \log(e) = e - 1 < 3 \\
  f(e^2) &= e^2 - \log(e^2) = e^2 - 2 > 4 < 9 \\
  f(e^3) &= e^3 - 3 > 3
\end{align*}
\]

* \( f(x) \) is cont. on \([e^2, e^3]\)

* \( f(e^1) < 3 = f(e^3) \)

So, by IVT, there exists \( \epsilon < c < e^3 \) s.t. \( f(c) = 3 \) (i.e. \( \log(c) = c - 3 \)).
Back to $\sqrt{2}$: Find an interval of length $\frac{1}{10}$ which contains $\sqrt{2}$.

Solution:

$$f(x) = x^2 - 2, \quad L = 0$$

\[\sqrt{\frac{1}{4}} \quad \sqrt{\frac{1}{4}} \quad 3/2\]

\[\frac{5}{4} \quad 1.375 \quad 1.4375\]

\[f(\phi) < 0 \quad \frac{1}{4} \quad f\left(\frac{3}{2}\right) = \frac{1}{4}\]

\[\frac{1}{4}\]

\[f(1.4375) > 0\]

\[1.375 < \sqrt{2} < 1.4375\]
Food for thought: Final 2010 (MATH 100) – Two points on the surface of the Earth are called antipodal if they are at exactly opposite points (for example, the North Pole and South Pole are antipodal points). Prove that, at any given moment, there are two antipodal points on the equator with exactly the same temperature.
Intro to Derivatives:

$$h(t) = -5t^2 + 30t \text{ m/s}$$

The height of the stone at moment $t$ (seconds).

**Question:** What is the velocity of the stone at time $t = 2$ sec?

**Average velocity:**

$$v_{av}(t) = \frac{h(t) - h(2)}{t - 2}$$

**Instantaneous velocity:**

$$v_{ins}(2) = \lim_{t \to 2} \frac{h(t) - h(2)}{t - 2} = 10 \text{ m/s}$$
Measuring the instantaneous rate of change of a function:

Definition: The derivative of the function $f(x)$ at $a$ is defined by:

$$\frac{df}{dx}(a) = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

(if the limit exists).

Remarks:

1. If $f'(a)$ exists, we say that $f(x)$ is differentiable at $a$. Otherwise, we say that $f(x)$ is non-differentiable.

2. If $f(x)$ is diff. at any $x$ in $(a,b)$, then $f'(x)$ is a function on $(a,b)$. 
Assume:
\[
\frac{f(x) - f(a)}{x - a} \quad x \to a \quad f'(a)
\]

average change in \( f(x) \).

Always: \( x - a \to 0 \)

so if \( f'(a) \) exists, then \( f(x) - f(a) \to 0 \)

i.e. \( f(x) \to f(a) \) \( x \to a \)

Theorem: If \( f(x) \) is diff. at \( a \),

then \( f(x) \) is cont. at \( a \).

So for \( x \) near \( a \),

\[
\frac{f(x) - f(a)}{x - a} \approx f'(a)
\]

\[
f(x) \approx f'(a) \cdot (x-a) + f(a)
\]

It's an approx.
not an equality.
The tangent of $f(x)$ at $x = a$

is

$$\ell(x) = f'(a) \cdot (x-a) + f(a)$$

This is the line through $(a, f(a))$ with slope $f'(a)$.

$$\ell(h) = f'(a) \cdot h + f(a)$$
Figure 3.1
Figure 3.8

0 > \lim_{\text{instantaneous rate of change}} = m_{\text{instantaneous rate of change}}

0 = \lim_{\text{instantaneous rate of change}} = m_{\text{instantaneous rate of change}}

m_{\text{derivative at } q} = \lim_{\text{instantaneous rate of change}} = m_{\text{derivative at } q}
Figure 3.9
Yesterday:
- IVT
- Intro to derivatives.

Today:
- Why wouldn't a function be diff.?
- First steps in differentiation.

Recall:
\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \]

* Tangent at \((a, f(a))\):
  \[ l(x) = f'(a) \cdot [x - a] + f(a) \]
  \[ l(x+h) = f'(a) \cdot h + f(a) \]
  \[ l(x) \approx f(x) \text{ for } x \text{ near } a \]
  \* \(f'(a)\) exists \(\Rightarrow f(x)\) is cont. at \(a\)

Warm-up problem: Calculate \(f'(4)\), where \(f(x) = \sqrt{x}\).

Solution:
\[ f'(4) = \lim_{x \to 4} \frac{f(x) - f(4)}{x - 4} = \lim_{x \to 4} \frac{\sqrt{x} - \sqrt{4}}{x - 4} \]
\[
\frac{\sqrt{x} - \sqrt{4}}{x - 4} = \frac{\sqrt{x} - \sqrt{4}}{x - 4} \cdot \frac{\sqrt{x} + \sqrt{4}}{\sqrt{x} + \sqrt{4}}
\]

\[
= \frac{x - 4}{(x - 4)(\sqrt{x} + \sqrt{4})} = \frac{1}{\sqrt{x} + \sqrt{4}} \quad x \to 4 \frac{1}{2\sqrt{4}} = \frac{1}{4}
\]

\[
f'(4) = \frac{1}{4}
\]

Example (Final 2010): If a function \( y = f(x) \) is differentiable at \( x = 3 \) and \( f'(3) = 5 \), find the limit \( \lim_{x \to 3} \frac{x^2 - 3x}{f(x) - f(3)} \).

Solution: \( \frac{5}{5} = f'(3) = \lim_{x \to 3} \frac{f(x) - f(3)}{x - 3} \)

\[
= \frac{x^2 - 3x}{f(x) - f(3)} = \frac{x(x - 3)}{x - 3} = -x \cdot \frac{x - 3}{f(x) - f(3)}
\]

\[
= \frac{f(x) - f(3)}{x - 3} \quad x \to 3 \frac{-3}{5} = \frac{f'(3)}{5}
\]

\[
a = \frac{1}{b} = \frac{b}{a}
\]
Why would a function not be diff. (at a)

1. If $f(x)$ is not cont. at $a$.
2. $f(x)$ has a knee at $a$.
   ("tangent from left" has different slope than "tangent to right")
3. $f(x)$ has a cusp at $a$.
   = the tangent (possibly "one sided tangent"
   is vertical.)
If $f$ is not continuous at $a$, then $f(x) = \lambda$ at $x = a$. 

$$f(x) = \lambda$$
Figure 3.19

\( x \)

\( f \)

\( \lambda \)

\( p \)

\( O \)

\( f(\lambda) = \lambda \)

\( p \leftarrow x \)

Tangents approach

\( f \)

Does not exist

Slope \( f \) does not exist

Slope \( f \) is not equal to slope \( f \)

Tangents approach
$\lim_{x \to 0^+} f'(x) = \infty$

$\lim_{x \to 0^-} f'(x) = -\infty$

$f'(0)$ does not exist.

$y = \sqrt{|x|}$

vertical line
tangent line

Figure 3.20 (a)
\[ \infty = (x), \lim_{x \to 0^-} x, \lim_{x \to 0^+} x \]

Figure 3.20 (b)
How to find the derivative?

Examples:
1. \( f(x) = C \)

\[
P'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{C - C}{x - a} = \lim_{x \to a} 0 = 0
\]

Theorem 3.2: \( \frac{d}{dx} (C) = 0 \).

O' (Final 2010) Determine whether the following statement is true or false:
If \( f(x) = g'(x) \), then \( f(x) = g(x) \).
FALSE: For any \( C \), let \( g(x) = f(x) + C \) then \( f'(x) = g'(x) \).
The converse is also true. If \( f'(x) = g'(x) \), then there is a constant \( C \) such that \( f(x) = g(x) + C \).
2. Linear functions:
\[ f(x) = x \]

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{x - a}{x - a} = 1 \]

Conclusion: \( \frac{d}{dx}(x) = 1 \).

2. \( f(x) = Ax + B \)

\[ f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

\[ = \lim_{x \to a} \frac{(Ax + B) - (Aa + B)}{x - a} \]

\[ = \lim_{x \to a} \frac{A(x - a)}{x - a} = A \]

Conclusion: \( \frac{d}{dx}(Ax + B) = A \).
Power rule:

\[ f(x) = x^n \quad n = \text{positive integer} \]

\( f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \)

\[ = \lim_{x \to a} \frac{x^n - a^n}{x - a} = \lim_{x \to a} (x-a)(x^2 + xa + a^2) \]

\[ = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} (x^2 + xa + a^2) = 3a^2. \]

Conclusion: \( \frac{d}{dx} (x^2) = 2x \).

\[ f(x) = x^3 \]

\[ f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \]

\[ = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} (x^2 + xa + a^2) \]

\[ = \lim_{x \to a} \frac{x^3 - a^3}{x - a} = \lim_{x \to a} (x^2 + xa + a^2) = 3a^2. \]

Theorem 3.3: \( \frac{d}{dx} (x^n) = n \cdot x^{n-1} \).
4) Constant Multiple Rule:

\[ f(x) = C \cdot g(x) \]

\[ f'(x) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{x \to a} \frac{Cg(x) - Cg(a)}{x - a} \]

\[ = C \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = Cg'(a) \]

\[ g'(a) \]

Theorem 3.4: \[ \frac{d}{dx} (C \cdot f(x)) = C \cdot \frac{df}{dx} \]

5) Sum Rule:

\[ (f + g)'(x) = \lim_{x \to a} \frac{(f + g)(x) - (f + g)(a)}{x - a} \]

\[ = \lim_{x \to a} \frac{f(x) + g(x) - f(a) - g(a)}{x - a} \]

\[ = \lim_{x \to a} \left[ \frac{f(x) - f(a)}{x - a} + \frac{g(x) - g(a)}{x - a} \right] \]
\[
\lim_{x \to a} \frac{f(x) - f(a)}{x - a} + \lim_{x \to a} \frac{g(x) - g(a)}{x - a} = f'(a) + g'(a)
\]

**Theorem 3.5:** \[
\frac{d}{dx} (f + g) = \frac{df}{dx} + \frac{dg}{dx}
\]

**Example:** \( f(x) = 3x^8 + 5x^6 + 4x^5 - 3x^2 + 1 \)

\[
f'(x) = \left[3x^8 + 5x^6 + 4x^5 - 3x^2 + 1\right]' \\
= \left[3x^8\right]' + \left[5x^6\right]' + \left[4x^5\right]' - \left[3x^2\right]' + \left[1\right]' \\
= 3 \cdot 8x^7 + 5 \cdot 6x^5 + 4 \cdot 5x^4 - 3 \cdot 2x \\
= 3 \cdot 8x^7 + 5 \cdot 6x^5 + 4 \cdot 5x^4 - 3 \cdot 2x \\
= 24x^7 + 30x^5 + 20x^4 - 6x.
\]
6. **Exponents**

\[ p(x) = b^x, \quad b > 0 \]

\[ p'(x) = \lim_{x \to a} \frac{p(x) - p(a)}{x - a} = \lim_{x \to a} \frac{b^x - b^a}{x - a} \]

\[ = \lim_{x \to a} \frac{b^a (b^{x-a} - 1)}{x - a} = b^a \left[ \lim_{x \to a} \frac{b^{x-a} - 1}{x - a} \right] = b^a \cdot p'(0) \]

**Fact:** \( p'(0) = \log(b) \)

**Theorem** \( \frac{d}{dx} (b^x) = b^x \cdot \log(b) \).

(8) \( \frac{d}{dx} (e^x) = e^x \)
Example: A germ culture grows with time. The number of germs in the culture at time $t$ (in hours) is given by

$$f(t) = e^{t+3} + 10,000.$$ 

Find the moment $t$ when the number $f(t)$ grows at the rate of 1,000,000 germs per hour.

Solution: Want to solve: $f'(t) = 1,000,000$.

$$f'(t) = ?$$

$$f'(t) = \left[ e^{t+3} + 10,000 \right]' = \left[ e^{t+3} \right]' + \left[ 10,000 \right]'$$

$$= \left[ e^{t+3} \right]' = e^3 \cdot \left[ e^{t+3} \right]' = e^3 \left[ (e^t)' \right]'$$

$$= e^3 \cdot (e^t)^t \cdot \log(e^t) = te^{t+3}$$

$$te^{t+3} = 1,000,000 \Rightarrow t = \frac{\log(1,000,000)}{3} - 3 \text{ hours.}$$
Product Rule:

What is \((f(x) \cdot g(x))'\)?

First guess: \(f'(x) \cdot g'(x)\) (No!)

Example: A rectangle has edges of length \(L_1(t) = 2 + 3t\) meters and \(L_2(t) = 3t\) (seconds).

The area of the rectangle is \(A(t) = L_1(t) \cdot L_2(t)\)

\[= 6t^2.\]

The rate of change of the area:

\[
\frac{dA}{dt} = \frac{d(6t^2)}{dt} = 6 \cdot \frac{d(t^2)}{dt} = 6 \cdot (2t) = 12t
\]

\[= 2 \cdot 3 = \frac{dL_1}{dt} \cdot \frac{dL_2}{dt}\]
\[ \frac{dL_1}{dt} \cdot L_2 + L_1 \cdot \frac{dL_2}{dt} = 2 \cdot 3t + 2t \cdot 3 = 12t \]

**Theorem:** \[ \frac{d}{dx} (f \cdot g) = \frac{df}{dx} \cdot g + f \cdot \frac{dg}{dx} \]

\[ (f \cdot g)' = f'g + f \cdot g' \]

**Example:**

1. \[ \frac{d}{dx} (x^2 \cdot e^x) \]

\[ = \frac{d}{dx}(x^2) \cdot e^x + x^2 \cdot \frac{d(e^x)}{dx} \]

\[ = 2x \cdot e^x + x^2 \cdot e^x = (2x + x^2) \cdot e^x \]

2. \[ \frac{d}{dt} \left( e^{t^2 + 3} \right) = e^3 \frac{d}{dt} \left( e^{t^2} \right) \]

\[ = e^3 \frac{d}{dt} \left( (e^t)^2 \right) = e^3 \cdot \left[ 2(e^t) \right] e^t \]

\[ \frac{e^t \cdot e^t \cdot e^t ... e^t}{3 \text{ times}} = \frac{e^t^{3+2}}{3} \]
Corollary: \( \frac{d}{dx} (f(x)^n) = n \cdot f(x)^{n-1} \cdot f'(x) \)

\[
\int f(x)^2 \, dx = \left[ f(x) \cdot f(x) \right] \\
= f'(x) \cdot f(x) + f(x) \cdot f'(x) = 2f(x) \cdot f'(x)
\]

\[
\int f(x)^3 \, dx = \left[ f(x)^2 \cdot f(x) \right] \\
= \left[ 2f(x) \cdot f'(x) \right] f(x) + f(x)^2 \cdot f'(x) = 3f(x)^2 \cdot f'(x)
\]