Exponential Models:

Ex: The amount of a radioactive isotope in a sample at time $t$ (in years) is given by $Q(t) = Q_0 e^{-kt}$, $k > 0$.

1. The half-life of the isotope is the $T_{\frac{1}{2}}$ s.t. $Q(T_{\frac{1}{2}}) = \frac{1}{2} Q_0$.

If $T_{\frac{1}{2}} = 30$ years, find $k$.

Sol: $\frac{1}{2} Q_0 = Q(30) = Q_0 e^{-k \cdot 30}$

$e^{-k \cdot 30} = \frac{1}{2}$

$log$ $\rightarrow$ $-k \cdot 30 = log(\frac{1}{2}) = -log(2)$

$k = \frac{log(2)}{30}$
Final Exam — December 18th 2015  Duration: 2.5 hours
This test has 11 questions on 14 pages, for a total of 100 points.

- Read all the questions carefully before starting to work.
- Q1-Q6 are short-answer questions[3pts each]; put your answer in the boxes provided.
- All other questions are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- Please circle your course and section:

   E - 25% - 30%
   M - 35% - 40%
   H - 30% - 40%


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Student Conduct during Examinations

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   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s), electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

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2 If the sample contained 20 "units" at 2000, how many "units" are in the sample now?

\[ t = 0 \rightarrow 2000 \]

\[ t = 18 \rightarrow 2018 \ (\text{now}) \]

\[ Q_0 = Q(0) = 20 \]

\[ Q(18) = Q_0 e^{-k \cdot 18} = 20 \cdot e^{-\frac{\log(2)}{18} \cdot 18} = 20 \cdot e^{-\frac{\log(2)}{18}} \]

\[ = 20 \left( e^{\log(2)} \right)^{-\frac{1}{18}} = 20 \cdot 2^{-\frac{1}{18}} = \frac{20}{\sqrt[18]{2}} \]

3 How fast is the isotope decaying? (now)

\[ \text{Sol: } Q'(18) = \frac{-\log(2) \cdot 20}{18} \]

\[ Q'(t) = -k \cdot Q_0 e^{-k \cdot t} = -k \cdot Q(t) \]
10. Let \( f(x) = x^2 + x - \frac{1}{x} - 1 \). Its derivative satisfies

\[
f'(x) = 2x + 1 + \frac{1}{x^2} = \frac{2x^3 + x^2 + 1}{x^2} = \frac{(x+1)(2x^2 - x + 1)}{x^2}.
\]

(a) Evaluate \( \lim_{{x \to 0^-}} f(x) \) and \( \lim_{{x \to 0^+}} f(x) \).

\[
\lim_{{x \to 0^-}} (x^2 + x - \frac{1}{x} - 1) = \lim_{{x \to 0^+}} (-\frac{1}{x}) = -\infty \quad \lim_{{x \to 0^-}} f(x) = +\infty
\]

(b) Find the intervals on which \( f \) is increasing or decreasing.

\[
CP: \quad (x+1)(2x^2 - x + 1) = 0 \\
\begin{align*}
  &x = -1 \\
  &x = 1 \pm \sqrt{1 - \frac{1}{2}}
\end{align*}
\]

\[
\text{Inc: } (-\infty, -1) \quad \text{Dec: } (-1, 0) \quad \text{Inc: } (0, \infty)
\]

(c) Find the intervals on which \( f \) is concave up. Hint: think about what version of \( f'(x) \)

\[
\text{PIP: } \quad f''(x) = 2 - \frac{1}{x^2} \quad x^3 = \frac{1}{3} \quad x = \frac{1}{\sqrt[3]{3}}
\]

\[
\text{CO: } (-\infty, 0), \left(\frac{1}{\sqrt[3]{3}}, \infty\right) \quad \text{CD: } (0, \frac{1}{\sqrt[3]{3}})
\]

(d) Sketch the graph of \( f \). Note: Your graph should be reasonably accurate within the grid, so you may not use the whole x-axis).
Let's check that $-1$ is a local minimum using the 2nd der. test.

$$f''(-1) = 2 - \frac{1}{(-1)^3} = 3 > 0$$

$\Rightarrow$ $-1$ is a loc. min. pt.
11. Consider the function \( f(x) = \frac{x^2 - 2}{(x - 2)^2} \), and its derivatives

\[
 f'(x) = \frac{4(1 - x)}{(x - 2)^3}, \quad f''(x) = \frac{4(2x + 1)}{(x - 2)^4}.
\]

3 marks  
(a) Find all the asymptotes of \( f(x) \)

\[
\lim_{x \to 2^-} f(x) = +\infty \quad \lim_{x \to 2^+} f(x) = -\infty
\]

HA: \( \lim_{x \to \pm \infty} \frac{x^2 - 2}{(x - 2)^2} = \frac{1}{1} = 1 \)

3 marks  
(b) Find the intervals on which \( f(x) \) is increasing or decreasing and classify the local extreme values.

\[
P_{\text{EP}}: \quad f'(x) = 0 \quad \Rightarrow \quad x = 1
\]

Inc: \( [1, \infty) \)

Dec: \( (-\infty, 1) \cup (2, \infty) \)

Min: \( x = 1 \)

3 marks  
(c) Determine where \( f(x) \) is concave up or down and find the inflection points.

\[
P_{\text{IPP}}: \quad f''(x) = 0 \quad \Rightarrow \quad x = -\frac{1}{2}
\]

CU: \( [-\frac{1}{2}, 2), (2, \infty) \)

CD: \( (-\infty, -\frac{1}{2}] \)

Inf: \( x = -\frac{1}{2} \)

3 marks  
(d) Sketch the graph of \( y = f(x) \).
Final 2013: Find the slope of the tangent line to \( y = x^3 + 3x^2 + 2 \) at its inflection point.

\[ y' = 3x^2 + 6x \]

PIP \( y'' = 6x + 6 \neq 0 \Rightarrow x = -1 \)

\[ y'' < 0 \quad \text{at} \quad x = -1 \]

\[ y(-1) = (-1)^3 + 3(-1)^2 + 2 = 4 \]

\[ y'(-1) = 3(-1)^2 + 6(-1) = -3 \]

\[ l(x) = -3(x - (-1)) + 4 = -3(x + 1) + 4 \]
Final Examination — December 14       Duration: 2.5 hours

This test has 13 questions on 18 pages, for a total of 100 points.

- Q1-Q8 are short-answer questions worth 48 pts [3 pts each]; put your answer in the boxes provided.
- Q9-Q13 are long-answer and worth 52 pts; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
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- PRINT your name and ID # very clearly. Failure to do so may result in a grade of 0:

First Name: ___________  Last Name: ___________

Student-No: ___________  Signature: ___________

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Review
6. (a) Below is a graph of \( f(x) \).

i. What is the global maximum value?

\[ x = 4, \quad y = 3 \]

Answer: 3

ii. State the \( x \)-value(s) where a local minimum occurs.

Answer: 2, 7

(b) The graph of \( f''(x) \) (the second derivative of \( f(x) \)) is displayed below. For what \( x \)-value(s) does \( f(x) \) have an inflection point?

\[ \text{PIP: } x = 0, 4, 8 \]

Answer: 0, 8
Short-Answer Questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given for work shown. Unless otherwise stated, calculator ready answers are acceptable.

3 marks 1. (a) Evaluate \( \lim_{x \to 7} \frac{2x^2 + 11x - 21}{x^2 + 8x + 7} \).

\[
\frac{2x^2 + 11x - 21}{x^2 + 8x + 7} = \frac{(x+7)(2x-3)}{(x+7)(x+1)} = \frac{2x-3}{x+1} \xrightarrow{x \to 7} -\frac{14-3}{7+1} = -\frac{11}{6} = \frac{11}{6}.
\]

Answer: \( \frac{11}{6} \)

\[
x^2 + 8x + 7 = A(x+1) + B(x+7) + C(x+7)^2
\]

\[
(x^2 + 8x + 7)' = 2x + 8
\]

\[
= -6(x+7) + (x+7)^2 = (x+7)(x+7-6) = (x+7)(x+1)
\]

\[
A = 0
\]

\[
B = -6
\]

\[
C = \frac{9}{2} = 1
\]

3 marks (b) How long would it take an investment of $100,000 with an interest rate of 6% compounded continuously to gain $8000?

Answer:
**Short-Answer Questions.** Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given for work shown. Unless otherwise stated, calculator ready answers are acceptable.

3 marks 1. (a) Compute \( \lim_{x \to -2} \frac{2x^2 - 5x - 18}{x + 2} \).

\[
2x^2 - 5x - 18 = (x+2)(2x-9)
\]

\[
\frac{2x^2 - 5x - 18}{x+2} = \frac{(x+2)(2x-9)}{x+2} = 2x-9 \quad x \neq -2
\]

Answer: \(-13\)

3 marks (b) Suppose \( f(2) = 3 \) and \( f'(2) = 2 \). Let \( h(x) = \frac{f(x)}{x^2} \). Find the equation of the line tangent to \( y = h(x) \) at \( x = 2 \).

\[
h(2) = \frac{f(2)}{2^2} = \frac{3}{4}
\]

\[
h'(x) = \frac{f'(x) \cdot x^2 - f(x) \cdot 2x}{x^4} = \frac{f'(x) \cdot x - 2f(x)}{x^3}
\]

\[
h'(2) = \frac{f'(2) \cdot 2 - 2 \cdot f(2)}{2^3} = \frac{0 \cdot 2 - 2 \cdot 3}{8} = -\frac{2}{8} = -\frac{1}{4}
\]

Answer:

\[
y = -\frac{1}{4}(x-2) + \frac{3}{4}
\]

3 marks (c) Compute the derivative of \( f(x) = \sin^2(e^x) \).

Answer:
4. (a) Compute the 3rd degree Taylor polynomial of \( f(x) = e^{-3x} \) at \( a = 0 \).

Answer:

(b) Determine an equation of the line tangent to the curve

\[ x^4 - x^2y + y^4 = 1 \]

at the point \((-1, 1)\).

\[
4x^3 - (2xy + x^2y') + 4y^3y' = 0
\]

Plug \( x = -1, y = 1 \):

\[
-4y - (-2 + y'(-1)) + 4y'(-1) = 0
\]

\[
3y'(-1) = 2 \quad \Rightarrow \quad y'(-1) = \frac{2}{3}
\]

(c) When a company manufactures \( x \) desks, the per-desk average cost is given by the function

\[
\overline{C}(x) = \frac{50}{x} + \frac{7}{\sqrt{x}} - \frac{100}{x^2}.
\]

Use this information to determine the marginal cost function.

Answer:
Full-solution problems - 10 pts each, except #11 which is worth 12pt: Justify your answers and show all your work for problems 9 and 13. Place a box around your final answer. Unless otherwise indicated, simplification of answers is required in these questions to earn full credit.

9. Consider the following function:

\[ f(x) = \begin{cases} 
  e^x, & x \leq 0 \\
  ax + b, & 0 < x \leq 1 \\
  1 + \sqrt{x}, & x > 1.
\end{cases} \]

(a) Determine values for \( a \) and \( b \) for which \( f(x) \) is continuous.

**Answer:**

\[ \begin{align*}
  b &= 1, \\
  a &= 1.
\end{align*} \]

(b) Determine whether or not \( f(x) \) can be differentiable at \( x = 0 \) or \( x = 1 \).

**Answer:**

Not diff. at \( x = 1 \), Yes diff. at \( x = 0 \).
8. (a) The graph on the left is the derivative of \( y = f'(x) \). Given that \( f(0) = 0 \), sketch the graph of \( f(x) \) on the right grid. Clearly label any local extrema and inflection points.

\begin{align*}
\begin{array}{c}
\text{Loc. ext.:} \\
\text{Loc. max.} \quad x = 2 \\
\text{Loc. min.} \quad x = 4 \\
\text{Inf.} \quad x = 1, 3
\end{array}
\end{align*}

(b) Consider the following function:

\[ f(x) = \begin{cases} 
4 - x^2, & x < 1 \\
\ln x, & 1 < x < 3 \\
e^x, & x > 3.
\end{cases} \]

Evaluate the limit

\[ \lim_{x \to 1^-} f(f(x)). \]

Answer:

\[ e^3 \]
4. (a) Use linear approximation to estimate $\sqrt{69}$.

Tangent of $f(x) = \sqrt{x}$ at $64$:

$f(64) = 8$

$f'(x) = \frac{1}{2\sqrt{x}} = \frac{1}{2} x^{-\frac{1}{2}}$

$f'(64) = \frac{1}{16}$

$y = \frac{1}{16}(x-64) + 8$

$\sqrt{69} \approx \frac{1}{16}(69-64) + 8 = \frac{5}{16} + 8$

(b) Decide whether the estimate from 5(a) is an over or under estimate and then use the formula, $M|x - a|$, to determine the worst case error.

$\varepsilon = \frac{1}{2} |x - a|$

$M = \frac{1}{2x^{3/2}}$

$\Rightarrow f(x)$ is CD

$|f''(x)| = \frac{1}{2x^{3/2}}$

$\leq \frac{1}{2 \cdot 64^{3/2}}$ per $x$ in $[64, 69]$
3 marks 3. (a) A (spherical) balloon is being deflated at the rate of 8 cm³/s. How fast is its radius changing when the radius is 2 cm? Note: The volume V of a sphere of radius r is \( \frac{4}{3} \pi r^3 \).

\[
\frac{dV}{dt} = -8 \left( \frac{cm^3}{s} \right)
\]

\[
\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{1}{4\pi r^2} \quad \Rightarrow \quad -8 = \frac{dV}{dt} \cdot \frac{1}{4\pi (2)^2}
\]

Answer:

3 marks (b) Find the value of \( a \) for which the function \( f(x) \) is continuous for all \( x \).

\[
f(x) = \begin{cases} 
  x^2 + a & \text{when } x \leq e, \\
  3a \ln(x) & \text{when } x > e.
\end{cases}
\]

Answer:

3 marks (c) Sketch a graph of a continuous function \( y = f(x) \) that satisfies the following:

- \( f(-5) = f(0) = 0 \);
- \( f'(x) < 0 \) for \(-5 < x < -3\) and \(1 < x < 5\);
- \( f'(x) > 0 \) for \(-3 < x < 1\);
- \( f''(x) < 0 \) for \(-2 < x < 3\);
- \( f''(x) > 0 \) for \(-5 < x < -2\) and \(3 < x < 5\);

Clearly label and local extrema and points of inflection.
11. A spotlight on the ground shines on a wall 15 m away. If a woman 2 m tall walks from the spotlight toward the wall at a speed of 0.7 m/s, how fast is the length of her shadow (on the building) changing when she is 8 m from the building? State your answer accurate to 2 decimal places.

Answer:
\[ \frac{dH}{dt} \bigg|_{t=3} = -\frac{3}{7} \text{ m/s} \]
13. A bucket is 60 cm high, has a radius of 40 cm at the top and 10 cm at the bottom. Water is being dumped into the bucket at a rate of 1 L/min. How fast is the water level rising when the water is 30 cm deep? You may use the fact that the volume of a truncated cone with height $h$, radius at the bottom $r_1$ and radius at the top $r_2$, is given by $V = \frac{1}{3}\pi h(r_2^2 + r_1 r_2 + r_1^2)$.

\[ \frac{dV}{dt} = +1 \quad \frac{h}{\text{min}} = +1000 \text{ cm}^3/\text{min} \]

\[ V = \frac{1}{3}\pi h (10^2 + 10 \cdot 5 + 5^2) \]

\[ \frac{r-10}{h-0} = \frac{40-10}{60-0} = \frac{30}{60} = \frac{1}{2} \]

\[ r = \frac{1}{2}h + 10 \]

\[ V = \frac{1}{3}\pi \left[ (10^2 + 10 \cdot 5 + 5^2) + h (10 + 25) \right] \]

\[ \frac{dV}{dt} = \frac{1}{3}\pi \left[ \frac{dh}{dt} (10^2 + 10 \cdot 5 + 5^2) + h (10 + 25) \right] \]

\[ h=30 \]

\[ r=25 \]

\[ \frac{dh}{dt} = \frac{1}{3}\pi \left[ \frac{dh}{dt} (10^2 + 10 \cdot 25 + 25^2) + 30 \cdot 60 \cdot \frac{1}{2} \frac{dh}{dt} \right] \]

\[ \frac{dh}{dt} \bigg|_{h=30} \]