Very short answer questions

1. Calculate the derivatives $f'(x)$ for the following functions:

(a) **2 marks** Compute $\frac{dy}{dz}$ where $\cos(yz) = 3yz$.

Solution: We have

\[
\frac{d}{dz} \cos(yz) = -\sin(yz) \frac{d(yz)}{dz} = -\sin(yz) (y + zy')
\]

And so

\[-\sin(yz) (y + zy') = 3y + 3y'z.
\]

Solving for $y'$ gives:

\[y' = \frac{-3y + y \sin(yz)}{z \sin(yz) + 3}
\]

(b) **2 marks** $f(x) = \frac{(x-3)\sqrt{x^2+2x}}{(x^4-4) \log(x)}$ DO NOT SIMPLIFY

Solution: Write

\[
\log(f(x)) = \log \left( \frac{(x-3)\sqrt{x^2+2x}}{(x^4-4) \log(x)} \right)
\]

\[= \log(x-3) + \log \sqrt{x^2+2x} - \log(x^4-4) - \log(\log(x))
\]

\[= \log(x-3) + \frac{1}{2} \log(x^2+2x) - \log(x^4-4) - \log(\log(x))
\]

Deriving both sides yields

\[
\frac{f'(x)}{f(x)} = \frac{1}{x-3} + \frac{1}{3} \frac{2x+2}{x^2+2x} - \frac{4x^3}{x^4-4} - \frac{1}{x \log(x)}.
\]

So

\[f'(x) = \left( \frac{1}{x-3} + \frac{1}{3} \frac{2x+2}{x^2+2x} - \frac{4x^3}{x^4-4} - \frac{1}{x \log(x)} \right) f(x).
\]

(c) **2 marks** Evaluate $f'(x)$, where $f(x) = \sin \left( e^{x^2} \right)$.
Solution: We apply the chain rule twice to get:

\[ f'(x) = \cos(e^{x^2}) \cdot \frac{d}{dx}(e^{x^2})' \]
\[ = \cos(e^{x^2}) \cdot e^{x^2} \cdot (x^2)' \]
\[ = \cos(e^{x^2}) \cdot e^{x^2} \cdot 2x \]

Long answer questions - You must show your work

2. [4 marks] A kite is flying 30 meters above the ground when the wind starts to blow it away in a direction parallel to the ground at the rate of 3 \( \frac{m}{sec} \). At what rate must the string be let out when the length of string already let out is 60 meters?

Answer: \( \frac{\sqrt{2700}}{20} \) \( \frac{m}{sec} \)

Solution: We denote the horizontal distance between the kite and the person operating it by \( x(t) \) (measured in meters) and the actual distance by \( s(t) \). We have \( s(t_0) = 60 \). We already know that \( \frac{dx}{dt} = 3 \frac{m}{sec} \).

By the Pythagorean theorem we have \( x(t)^2 + 30^2 = s(t)^2 \). Differentiating this with respect to \( t \) yields \( 2x(t) \frac{dx}{dt} = 2s(t) \frac{ds}{dt} \). We note that \( x(t_0)^2 + 30^2 = s(t_0)^2 = 60^2 \) and hence \( x(t_0) = \sqrt{60^2 - 30^2} = \sqrt{2700} \). Plugging everything into \( 2x(t) \frac{dx}{dt} = 2s(t) \frac{ds}{dt} \) yields \( \frac{dx}{dt}(t_0) = \frac{x(t_0)}{s(t_0)} \frac{ds}{dt}(t_0) = \frac{\sqrt{2700}}{60} \cdot 3 = \frac{\sqrt{2700}}{20} \frac{m}{sec} \).