Midterm 1 Practice Test  
Duration: 1 hour  
This test has 6 questions on 7 pages, for a total of 50 points.

- Read all the questions carefully before starting to work.
- Q1-Q4 are short-answer questions [3 pts each]; put your answer in the boxes provided.
- Q5 and Q6 are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. **None of the following are allowed:** documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)
- Please circle your course and section:

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First Name: ___________________  Last Name: ___________________

Student-No: ___________________  Signature: ___________________

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Score: ___________________

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**Student Conduct during Examinations**

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBC card for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   - (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   - (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   - (iii) purposely viewing the written papers of other examination candidates;
   - (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and
   - (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) [electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing].

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
Short-Answer Questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given for work shown. Unless otherwise stated, calculator ready answers are acceptable.

1. (a) An investment of $35,000 grew to $37,000 over 18 months. What is the effective interest rate?

Answer: \[ i = \left( \frac{37}{35} \right)^{2/3} - 1 \]

Solution:

\[
FV = PV \left( 1 + \frac{i}{n} \right)^{nt} \\
37,000 = 35,000 \left( 1 + \frac{i}{12} \right)^{18/12} \\
\frac{37}{35} = \left( 1 + \frac{i}{12} \right)^{3/2} \\
\left( \frac{37}{35} \right)^{2/3} = 1 + i
\]

(b) How long would it take an investment of $500 to grow to $750 if the it was compounded semi-annually at the rate of 6%.

Answer: \[ \frac{\ln 3/2}{2 \ln 1.03} \]

Solution:

\[
FV = PV \left( 1 + \frac{i}{n} \right)^{nt} \\
750 = 500 \left( 1 + \frac{0.06}{2} \right)^{2t} \\
\frac{750}{500} = (1.03)^{2t} \\
\ln \left( \frac{3}{2} \right) = 2t \ln 1.03
\]

(c) Evaluate \( \lim_{n \to \infty} 3000 \left( 1 + \frac{0.07}{n} \right)^{3n/2} \).

Answer: \( 3000e^{0.07 \cdot 3/2} \)

Solution: We can interpret this as continuously compounded interest.
2. (a) Find the value of \( a \) for which the function \( f(x) \) is continuous for all \( x \).

\[
f(x) = \begin{cases} 
  x^2 + a & \text{when } x \leq e, \\
  3a \ln(x) & \text{when } x > e.
\end{cases}
\]

**Solution:** As the function is piecewise continuous, we just need to check for continuity at the point \( e \). This is the same as \( \lim_{x \to e^-} f(x) = \lim_{x \to e^+} f(x) = f(e) \). So, we have

\[
f(e) = \lim_{x \to e^-} f(x) = e^2 + a
\]

and:

\[
\lim_{x \to e^+} f(x) = 3a \ln e = 3a
\]

This tells us that \( e^2 + a = 3a \), and solving for \( a \) gives \( a = \frac{e^2}{2} \).

(b) Use the IVT to prove that the equation \( e^x = x^2 \) has a solution.

**Solution:** Define \( f(x) = e^x - x^2 \), which is continuous everywhere, so we just need to find a sign change. We have

1. \( f(0) = e^0 - 0^2 = 1 > 0 \) and
2. \( f(-1) = \frac{1}{e} - 1 < 0 \).

Therefore, the IVT guarantees that there exist a value \( c \) between \(-1\) and \( 0 \) such that \( f(c) = 0 \); in particular \( e^c = c^2 \).
3 marks 3. (a) Evaluate \( \lim_{x \to 3} \frac{2x^2 - 4x - 6}{x^2 + 4x - 21} \).

Answer: 4/5

**Solution:** Direct substitution yields 0/0, which tells us that \((x - 3)\) must be a factor of both the numerator and the denominator. So we should factor and cancel:

\[
\frac{2x^2 - 4x - 6}{x^2 + 4x - 21} = \frac{(2x + 2)(x - 3)}{(x + 7)(x - 3)} = \frac{2x + 2}{x + 7}
\]

Hence the limit is \( \lim_{x \to 3} \frac{2x^2 - 4x - 6}{x^2 + 4x - 21} = \lim_{x \to 3} \frac{2x + 2}{x + 7} = \frac{8}{10} = \frac{4}{5} \).

3 marks (b) Find the equation of the tangent line to \( f(x) = \frac{x^2 + 1}{e^x} \) at \( x = 1 \).

Answer: \( y = \frac{2}{e} \)

**Solution:** We’ll need the derivative:

\[
f(x) = \frac{x^2 + 1}{e^x}
\]

\[
f'(x) = \frac{2xe^x - e^x(x^2 + 1)}{(e^x)^2}
\]

\[
= \frac{2x - (x^2 + 1)}{e^x}
\]

Therefore, \( f'(1) = 0 \), which makes the line horizontal.

3 marks (c) Compute the derivative of \( f(x) = \sin^2(e^x) \).

Answer: \( 2 \sin(e^x) \cdot \cos(e^x) \cdot e^x \)

**Solution:** We just need to use the Chain Rule a couple of times.
3 marks 4. (a) Determine an equation of the line tangent to the curve

\[ x^4 - x^2y + y^4 = 1 \]

at the point \((-1, 1)\).

Answer: \( y = 1 + \frac{2}{3}(x + 1) \)

**Solution:** We need \( y' \) at the point \((-1, 1)\).

\[ x^4 - x^2y + y^4 = 1 \]
\[ 4x^3 - 2xy - x^2y' + 4y^3y' = 0 \]

Note that we can sub in our values here to get:

\[ 4(-1) + 2 - y' + 4y' = 0 \]
\[ 3y' = 2 \]
\[ y' = \frac{2}{3} \]

Therefore, the tangent line is \( y = 1 + \frac{2}{3}(x + 1) \)

3 marks (b) If a rock is thrown upward on a planet with an initial velocity of 10 m/s, its height above the ground level is given by

\[ h(t) = 10t - 4t^2. \]

Determine the rock’s velocity when it hits the ground.

Answer: -10 m/s

**Solution:** \( h(t) = 10t - 4t^2 \Rightarrow v(t) = \frac{dh}{dt} = 10 - 8t \). We need to know when the rock hits the ground. To get this, we set \( h(t) = 0 \)

\[ h(t) = 10t - 4t^2 \]
\[ 0 = t(10 - 4t) \]

From which we see that \( t = 0 \) or \( 10 - 4t = 0 \Rightarrow t = 10/4 = 5/2 \).

Therefore, we have \( v(2.5) = 10 - 8 \cdot \frac{5}{2} = 10 - 20 = -10 \text{ m/s} \).
Full-solution problems: Justify your answers and show all your work. Place a box around your final answer. Unless otherwise indicated, simplification of answers are required in these questions.

10 marks 5. Let $f(x) = \sqrt{2x^2 + 1}$. Use the definition of the derivative to find $f'(2)$. No marks will be given for the use of any differentiation rules.

Solution:

\[
f'(2) = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2} = \lim_{x \to 2} \frac{\sqrt{2x^2 + 1} - \sqrt{9}}{x - 2} = \lim_{x \to 2} \frac{\sqrt{2x^2 + 1} - 3}{x - 2} \cdot \frac{\sqrt{2x^2 + 1} + 3}{\sqrt{2x^2 + 1} + 3} = \lim_{x \to 2} \frac{2x^2 + 1 - 9}{x - 2} \cdot \frac{1}{\sqrt{2x^2 + 1} + 3} = \lim_{x \to 2} \frac{2x^2 - 8}{(x - 2) \cdot (\sqrt{2x^2 + 1} + 3)} = \lim_{x \to 2} \frac{2(x - 2)(x + 2)}{(x - 2) \cdot (\sqrt{2x^2 + 1} + 3)} = \lim_{x \to 2} \frac{2(x + 2)}{\sqrt{2x^2 + 1} + 3} = \frac{8}{6} = \frac{4}{3}
\]
6. A company is planning to produce a new electric toaster. After conducting an extensive market survey, the research department provides the following information: when the unit price is $16 per toaster, then the weekly demand is 20 toasters; for every $2 decrease in the unit price, the weekly demand increases by 10 toasters. The financial department estimate that the weekly fix costs will be $140 and the variable cost of production will be $4 per toaster. Let \( p \) be the unit price and let \( q \) be weekly demand.

Determine the price should the company charge in order to maximize the weekly profit?

**Answer:** $12

**Solution:** A data point is \((q, p) = (\text{quantity}, \text{price})\). So the demand curve will be given by \( p = mq + b \) and will have slope

\[
m = \frac{\text{rise}}{\text{run}} = \frac{\Delta p}{\Delta q} = \frac{-2}{10} = \frac{-1}{5}.
\]

So the equation becomes

\[
p = \frac{-1}{5} q + b.
\]

Substituting in the point \( q = 20 \) and \( p = 16 \) yields:

\[
16 = \frac{-1}{5} \cdot 20 + b
\]
\[
16 = -4 + b
\]
\[
20 = b
\]

Therefore,

\[
p = \frac{-1}{5} q + 20
\]

From here we see that \( R = \frac{-1}{5} q^2 + 20q \) and Profit will be given by:

\[
P = R - C = \frac{-1}{5} q^2 + 20q - (140 + 4q) = \frac{-1}{5} q^2 + 16q - 140
\]

To find its maximum we can find the vertex or set its derivative to 0.

\[
\frac{dP}{dq} = \frac{-2}{5}q + 16
\]

So that max profit will occur when \( \frac{-2}{5}q + 16 = 0 \Rightarrow q = 40 \), from which we see that price must be \(-8 + 20 = \$12\).