Midterm — June 4th 2018  Duration: 50 Minutes

This test has 5 questions on 9 pages, for a total of 40 points.

- Read all the questions carefully before starting to work.
- Q1-Q3 are short-answer questions; put your answer in the boxes provided.
- Q4 and Q5 are long-answer; you should give complete arguments and explanations for all your calculations; answers without justifications will not be marked.
- Continue on the back of the previous page if you run out of space.
- Attempt to answer all questions for partial credit.
- This is a closed-book examination. None of the following are allowed: documents, cheat sheets or electronic devices of any kind (including calculators, cell phones, etc.)

First Name: ______________________  Last Name: ______________________

Student-No: ______________________  Signature: ______________________

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Score: ______________________

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Student Conduct during Examinations

1. Each examination candidate must be prepared to produce, upon the request of the invigilator or examiner, his or her UBCcard for identification.

2. Examination candidates are not permitted to ask questions of the examiners or invigilators, except in cases of supposed errors or ambiguities in examination questions, illegible or missing material, or the like.

3. No examination candidate shall be permitted to enter the examination room after the expiration of one-half hour from the scheduled starting time, or to leave during the first half hour of the examination. Should the examination run forty-five (45) minutes or less, no examination candidate shall be permitted to enter the examination room once the examination has begun.

4. Examination candidates must conduct themselves honestly and in accordance with established rules for a given examination, which will be articulated by the examiner or invigilator prior to the examination commencing. Should dishonest behaviour be observed by the examiner(s) or invigilator(s), pleas of accident or forgetfulness shall not be received.

5. Examination candidates suspected of any of the following, or any other similar practices, may be immediately dismissed from the examination by the examiner/invigilator, and may be subject to disciplinary action:
   (i) speaking or communicating with other examination candidates, unless otherwise authorized;
   (ii) purposely exposing written papers to the view of other examination candidates or imaging devices;
   (iii) purposely viewing the written papers of other examination candidates;
   (iv) using or having visible at the place of writing any books, papers or other memory aid devices other than those authorized by the examiner(s); and,
   (v) using or operating electronic devices including but not limited to telephones, calculators, computers, or similar devices other than those authorized by the examiner(s) electronic devices other than those authorized by the examiner(s) must be completely powered down if present at the place of writing).

6. Examination candidates must not destroy or damage any examination material, must hand in all examination papers, and must not take any examination material from the examination room without permission of the examiner or invigilator.

7. Notwithstanding the above, for any mode of examination that does not fall into the traditional, paper-based method, examination candidates shall adhere to any special rules for conduct as established and articulated by the examiner.

8. Examination candidates must follow any additional examination rules or directions communicated by the examiner(s) or invigilator(s).
Short-Answer Questions. Put your answer in the box provided. Full marks will be given for a correct answer placed in the box, while part marks may be given for work shown. Unless otherwise stated, calculator ready answers are acceptable.

1. (a) You invest money in a fund compounded daily for two years with a nominal rate of 5%. You then switch to a fund compounded continuously with a real interest rate of 5% for another three years. At the end of this period the fund increased by $10,000, how much money did you invest initially?

\[
PV = \frac{10,000}{(1 + 0.05)^{365} e^{0.15} - 1}
\]

Solution: 
- \( PV \) - initial investment
- \( FV_1 \) - fund after the first two years
- \( FV_2 \) - fund after the extra three years

\[
FV_1 = PV \left( 1 + \frac{i}{n} \right)^{nt_1}, \quad i = 0.05, \quad n = 365, \quad t_1 = 2
\]

\[
FV_2 = FV_1 e^{rt_2}, \quad r = 0.05, \quad t_2 = 3
\]

\[
FV_2 = FV_1 e^{0.15} = PV \left( 1 + \frac{0.05}{365} \right)^{730} e^{0.15}
\]

\[
PV + 10,000 = FV_1 e^{0.15} = PV \left( 1 + \frac{0.05}{365} \right)^{730} e^{0.15}
\]

\[
PV = \frac{10,000}{(1 + \frac{0.05}{365})^{730} e^{0.15} - 1}
\]

(b) An umbrella factory sells 6,000 umbrellas a year at the cost of $8 a piece. An umbrella sale last December showed that a decrease of $1 in the price of an umbrella caused an increase in selling of 500 umbrellas a month.

i. Find the linear demand equation for the umbrellas.

\[
q = -500 \cdot p + 4,500
\]

or

\[
q = -500 \cdot p + 10,000
\]

Solution: 
An annual sell of 6,000 umbrellas is equivalent to a monthly sell of 500 units. The linear demand equation is given by \( q = A \cdot p + B \). The data we are given implies that

\[
500 = A \cdot 8 + B
\]

\[
1,000 = A \cdot 7 + B
\]

Subtracting the two equations yield \( A = -500 \). Plugging \( A \) into the first equation we get

\[
500 = -500 \cdot 8 + B
\]

and hence \( B = 4,500 \).

If you missed the fact that 6,000 umbrellas was the annual and not the monthly product: The linear demand equation is given by \( q = A \cdot p + B \).
The data we are given implies that

\[ 6,000 = A \cdot 8 + B \]
\[ 6,500 = A \cdot 7 + B \]

Subtracting the two equations yield \( A = -500 \). Plugging \( A \) into the first equation we get

\[ 6,000 = -500 \cdot 8 + B \]

and hence \( B = 10,000 \).

ii. Compute the maximal revenue of the factory.

| Answer: $10,125 
or $50,000 |

**Solution:** The revenue is given by \( R = p \cdot (-500 \cdot p + 4,500) \) whose zeros are at \( p = $0 \) and \( p = $9 \) and hence the maximal revenue will be \( 4.5 \cdot (4,500 - 500 \cdot 4.5) = $10,125 \) when the price is $4.5 a piece. In this case they sell 2,250 umbrellas a month.

**If you missed the fact that 6,000 umbrellas was the annual and not the monthly product:** The revenue is given by \( R = p \cdot (-500 \cdot p + 10,000) \) whose zeros are at \( p = $0 \) and \( p = $20 \) and hence the maximal revenue will be $50,000 when the price is $10 a piece. In this case they sell 5,000 umbrellas a month.
2. (a) Find a value $a$ so that the function

$$f(x) = \begin{cases} \sqrt{x^2 + a}, & x \leq a \\ x + a, & x > a \end{cases}$$

is continuous for all $x$.

\textbf{Solution:}

$$f(a) = \lim_{x \to a^-} f(x) = \lim_{x \to a^-} \sqrt{x^2 + a} = \sqrt{a^2 + a}$$

$$\lim_{x \to a^+} f(x) = \lim_{x \to a^+} x + a = 2a$$

In order for the limit to exist (and to equal the value of the function) we need the two one-sided limits to be equal to each other, namely

$$\sqrt{a^2 + a} = 2a$$

$$a^2 + a = 4a^2$$

$$3a^2 - a = 0$$

$$a(3a - 1) = 0$$

This equation has two solutions:

$$a = 0, \frac{1}{3},$$

both values will be accepted as the final answer.

\textbf{Alternative Solution:}

If $a = 0$, then $f(x) = |x|$ and hence it is continuous.

(b) Show that the equation $\ln(x) = x^3 - 2$ has a solution. Justify your answer.

\textbf{Solution:} We define the function $f(x) = \ln(x) - x^3$. We wish to find a point $c$ such that $f(c) = -2$. We have

$$f(1) = \ln(1) - 1^3 = -1 > -2$$

$$f(2) = \ln(2) - 2^3 = \ln(2) - 8 < 2 - 8 = -6 < -2.$$  

We also note that $f(x)$ is continuous on $[1, 2]$ since it is the sum of $\ln(x)$ (which is continuous there) and a polynomial (which is continuous at any $x$).

It follows that there exist $1 < c < 2$ such that $f(c) = -2$ and hence $\ln(x) = c^3 - 2$.

\textbf{Alternative Solution:} We have

$$f(e) = \ln(e) - e^3 = 1 - e^3 < 1 - 2^3 = -7 < -2.$$  

So on can use the interval $[1, e]$ instead of $[1, 2]$. There are of course many other intervals that could work.
3 marks 3. (a) Evaluate
\[ \lim_{x \to 1} \frac{4x^2 - 3x - 1}{x^2 - 3x + 2} \]

Answer: \(-5\)

Solution: We first check that we can’t just plug \(x = 3\) into the function
\[ 4 \cdot 1^2 - 3 \cdot 1 - 1 = 0, \quad 1^2 - 3 \cdot 1 + 2 = 0 \]
and hence both the numerator and denominator has a factor \(x - 1\). Factoring gives us
\[
\begin{align*}
4x^2 - 3x - 1 &= (x - 1)(4x + 1) \\
x^2 - 3x + 2 &= (x - 1)(x - 2)
\end{align*}
\]
and hence, for \(x \neq 1, 2\),
\[
\frac{4x^2 - 3x - 1}{x^2 - 3x + 2} = \frac{(x - 1)(4x + 1)}{(x - 1)(x - 2)} = \frac{4x + 1}{x - 2}.
\]

In particular
\[
\lim_{x \to 1} \frac{4x^2 - 3x - 1}{x^2 - 3x + 2} = \lim_{x \to 1} \frac{4x + 1}{x - 2} = \frac{4 \cdot 1 + 1}{1 - 2} = -5.
\]

3 marks (b) Compute the tangent line of \(y = \frac{4^x}{x^2 + 1}\) at \(x = 0\).

Answer: \(l(x) = (\ln 4)x + 1\)

Solution:
\[
\begin{align*}
y(0) &= \frac{4^0}{0^2 + 1} = 1 \\
y' &= \frac{(4^x \ln 4)(x^2 + 1) - (3x)2x}{(x^2 + 1)^2} \\
y'(0) &= \frac{(4^0 \ln 4) \cdot (0^2 + 1) - 4^0 \cdot 2 \cdot 0}{(0^2 + 1)^2} = \ln 4 \\
l(x) &= (\ln 4)(x - 0) + 1 = (\ln 4)x + 1
\end{align*}
\]

3 marks (c) Compute \(y'\) given \(y = e^{\log(x)^2 + \sin(x^2)}\)

Answer:
\[
e^{\log(x)^2 + \sin(x^2)} \cdot \left(\frac{2 \log(x)}{x} + 2x \cos(x^2)\right)
\]
Solution:

\[ f'(x) = e^{\log(x)^2 + \sin(x^2)} \cdot (\log(x)^2 + \sin(x^2))' \]

\[ = e^{\log(x)^2 + \sin(x^2)} \cdot (2 \log(x) \cdot \log(x)' + \cos(x^2) \cdot (x^2)') \]

\[ = e^{\log(x)^2 + \sin(x^2)} \cdot \left( \frac{2 \log(x)}{x} + 2x \cos(x^2) \right) \]
4. Consider the curve given by $y^2 - xy + x^3 = 1$. Find $f''(1)$ assuming that $y = f(x)$ near $(1, 0)$.

Answer: 18

Solution: We derive the equation once:

$$2yy' - y - xy' + 3x^2 = 0$$

and again:

$$2y^2 + 2yy'' - y' - y - xy'' + 6x = 0.$$

Simplifying, we get:

$$y'(x) = \frac{y - 3x^2}{2y - x}$$

$$y''(x) = \frac{2y' - 2y^2 - 6x}{2y - x}$$

Plugging $x = 1$ and $y = 0$ in the first equation yields

$$y'(0) = \frac{0 - 3 \cdot 1^2}{2 \cdot 0 - 1} = 3$$

and plugging $x = 1$, $y = 0$ and $y' = 3$ into the second equation yields:

$$y''(0) = \frac{2 \cdot 3 - 2 \cdot 3^2 - 6 \cdot 1}{2 \cdot 0 - 1} = \frac{-18}{-1} = 18.$$ 

Note that it is possible to first plug in $x = 1$, $y = 0$ and $y' = 3$ and then simplify.

Alternative Solution:

After getting $y'(x) = \frac{y - 3x^2}{2y - x}$ and $y'(1) = 3$ one can differentiate that to get

$$y''(x) = \frac{(y' - 6x)(2y - x) - (2y' - 1)y - 3x^2}{(2y - x)^2}$$

and plug $x = 1$, $y = 0$ and $y' = 3$ into that.
5. Let \( f(x) = \frac{1}{\sqrt{x+1}} \). Use the definition of the derivative to find \( f'(1) \). No marks will be given for the use of any differentiation rules. A bonus point will be granted to students who will calculate \( f'(a) \) for arbitrary \( a > -1 \).

**Solution:** By the definition of the derivative

\[
f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{2}}}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{2} - \sqrt{x+1}}{(x-1)\sqrt{(x+1)(2)}}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{2} - \sqrt{x+1}}{-(x-1)(\sqrt{\sqrt{2} + \sqrt{x+1}})\sqrt{(x+1)(2)}}
\]

\[
= \lim_{x \to 1} \frac{-1}{2\sqrt{2} \cdot (\sqrt{2})^2} = -\frac{1}{2^{5/2}}.
\]

**Alternative Solution:**

\[
f'(1) = \lim_{x \to 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \to 1} \frac{\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{2}}}{x - 1}
\]

\[
= \lim_{x \to 1} \frac{\sqrt{2} - \sqrt{x+1}}{(x-1)\left(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{2}}\right)}
\]

\[
= \lim_{x \to 1} \frac{2 - (x-1)}{2(x+1)(x-1)\left(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{2}}\right)}
\]

\[
= \lim_{x \to 1} \frac{1 - x}{2(x+1)(x-1)\left(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{2}}\right)}
\]

\[
= \lim_{x \to 1} \frac{-1}{2(x+1)(\frac{1}{\sqrt{x+1}} + \frac{1}{\sqrt{2}})}
\]

\[
= -\frac{1}{2 \cdot \left(\frac{2}{\sqrt{2}}\right)} = -\frac{1}{2^{5/2}}.
\]
For arbitrary $a > -1$:

\[
f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
= \lim_{x \to a} \frac{\frac{1}{\sqrt{x+1}} - \frac{1}{\sqrt{a+1}}}{x - a} \\
= \lim_{x \to a} \frac{\sqrt{a+1} - \sqrt{x+1}}{(x-a)(\sqrt{a+1} + \sqrt{x+1})(x+1)(a+1)} \\
= \lim_{x \to a} \frac{(a+1) - (x+1)}{(x-a)(\sqrt{a+1} + \sqrt{x+1})(x+1)(a+1)} \\
= \lim_{x \to a} \frac{-1}{(\sqrt{a+1} + \sqrt{x+1})(x+1)(a+1)} \\
= \frac{-1}{2\sqrt{a+1} \cdot \sqrt{(a+1)^2}} = \frac{-1}{2(a+1)^{3/2}}.
\]