Short answer questions — you must show your work

1. **2 marks** Evaluate \( f'(x) \), where \( f(x) = \frac{\cos x}{x^3+1} \).

   **Answer:**
   
   \[
   f'(x) = - \frac{(x^3+1) \sin x + 3x^2 \cos x}{(x^3+1)^2}
   \]

   **Solution:** We apply the quotient rule:
   
   \[
   f'(x) = \frac{(\cos x)'(x^3 + 1) - \cos x(x^3 + 1)'}{(x^3 + 1)^2}
   = \frac{- \sin x(x^3 + 1) - \cos x(3x^2)}{(x^3 + 1)^2}
   = \frac{-(x^3 + 1) \sin x + 3x^2 \cos x}{(x^3 + 1)^2}
   \]

2. **2 marks** Evaluate \( f'(x) \), where \( f(x) = \sin(e^{x^2}) \).

   **Answer:** \( f'(x) = 2x \cdot \cos(e^{x^2}) \cdot e^{x^2} \)

   **Solution:** We apply the chain rule twice to get:
   
   \[
   f'(x) = \cos(e^{x^2}) \cdot [e^{x^2}]'
   = \cos(e^{x^2}) \cdot e^{x^2} \cdot (x^2)'
   = \cos(e^{x^2}) \cdot e^{x^2} \cdot 2x
   \]

3. **2 marks** Evaluate \( f^{(2)}(x) \), where \( f(x) = \sin(x^3) \).

   **Answer:**
   
   \[
   f^{(2)}(x) = 6x \cos(x^3) - 9x^4 \sin(x^3)
   \]
Solution: We apply the chain rule to get the first derivative:

\[ f'(x) = \cos(x^3) \cdot (x^3)' = 3x^2 \cos(x^3). \]

In order to get the second derivative we apply the product and chain rules:

\[ f''(x) = f''(x) = (f')'(x) = 6x \cos(x^3) + 3x^2(-\sin(x^3))(x^3)' = 6x \cos(x^3) - 9x^4 \sin(x^3) \]

Long answer question — you must show your work

4. [4 marks] An interstellar spaceship travels from Alpha Centauri A to Epsilon Eridani. Due to galactic politics, its trajectory is given by the curve \(2(y - 2)^2 + x^2 = 1\). The pilot wishes to send a mail probe to Earth, located at the point \((0, 0)\). Given that the probe will fly along the tangent to the ship's trajectory at the point of departure, check if the probe will get to Earth assuming the pilot releases it at the point \((\sqrt{\frac{11}{4}}, \frac{7}{4})\).

Answer: Yes

Solution: In order to solve the question, we need to write the line-equation for the tangent of the curve at the given point and to check if it passes through the origin.

Deriving the equation \(2(y - 2)^2 + x^2 = 1\) from both sides yields

\[ 4(y - 2)y' + 2x = 0 \]

Namely

\[ y' = \frac{x}{2(2 - y)} \]

By plugging \(x = \sqrt{\frac{11}{4}}\) and \(y = \frac{7}{4}\) we get that \(y' = \frac{\sqrt{11}}{2(2 - \frac{7}{4})} = \frac{\sqrt{11}}{2}\).

Hence, the tangent to the curve at this point is given by

\[ y = \frac{\sqrt{11}}{2}(x - \frac{\sqrt{11}}{4}) + \frac{7}{4} = \frac{\sqrt{11}}{2}x - \frac{14}{8} + \frac{7}{4} = \frac{\sqrt{11}}{2}x, \]

which passes through \((0, 0)\).