Guidelines for Solving Optimization Problems

☐ Understand the problem.

☐ Draw a sketch (if applicable), introduce notations.

☐ Write equations, reduce to one variable.

☐ Solve the problem - apply calculus (Closed interval method, Absolute extreme theorem, concavity argument, graphical solution). Be warned that when the domain is not a closed interval, it is possible that there might not be an absolute extreme value!

☐ Reflect.
Question 3.
Find the point of the line $6x + y = 9$ that is closest to the point $(-3, 0)$.

Target function:

$D = \sqrt{(x - (-3))^2 + (y - 0)^2}$

We want to minimize the distance.

We can minimize $D$ or $D^2$ and since $D > 0$ it will be minimized at the same point.

$D^2 (x) = (x + 3)^2 + y^2 = (x + 3)^2 + (9 - 6x)^2$

for any $x$.

$\frac{dD^2}{dx} = 2(x + 3) + 2(9 - 6x) \cdot (-6) = 2x + 6 - 12(9 - 6x)$

$= 74x - 102 \equiv 0$

$x = \frac{102}{74} = \frac{51}{37}$ is the only critical point.
\[
\frac{d^2(D^2)}{dx^2} = (D^2)'' = 74 > 0
\]

\[\rightarrow x = \frac{51}{37} \text{ a local minimum.}\]

This only local extremum so its an absolute minimum.

\[x = \frac{51}{37}, \quad y = 9 - 6 \cdot \frac{51}{37} + \cdots\]
\[d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2\]

\[
\frac{d \left( \frac{d^2}{dt} \right)}{dt} = 2 \frac{d}{dt} \cdot d'(t)
\]
Question 4.
A cylindrical can is being made to contain 1 L of oil. Find the dimensions that will minimize the amount of metal needed to make the can.

Target function:

\[ V = \pi R^2 h = 1 \text{ L} = 1000 \text{ cm}^3 \]

\[ A = 2\pi R^2 + 2\pi Rh \]

\[ h = \frac{1000}{\pi R^2} \]

\[ A(R) = 2\pi R^2 + 2\pi R \cdot \frac{1000}{\pi R^2} = 2\pi R^2 + \frac{2000}{R} \]

\[ \frac{dA}{dR} = 4\pi R - \frac{2000}{R^2} = 0 \]

\[ 4\pi R = \frac{2000}{R^2} \rightarrow R^3 = \frac{500}{\pi} \rightarrow R = \left(\frac{500}{\pi}\right)^{\frac{1}{3}} \text{ cm} \]

\[ h = \frac{1000}{\pi \left(\frac{500}{\pi}\right)^{\frac{2}{3}}} \text{ cm} \]
Question 5.
If 1200 cm$^2$ of material is available to make a box with a square base and open top, find the largest possible volume of the box.

Target function: $V = x^2h$

$1200 \text{cm}^2 = A = x^2 + 4xh$

$h = \frac{1200 - x^2}{4x}$

$X > 0$
$h > 0$

$V(x) = x^2 \frac{1200 - x^2}{4x} = \frac{(1200 - x^2)x}{4} = \frac{1}{4}(1200x - x^3)$

$\frac{dV}{dx} = \frac{1}{4}(1200 - 3x^2) = 0$

$3x^2 = 1200$
$x^2 = 400$
$x = \pm 20$

$x = +20$

is the only critical point on $(0, \infty)$
\[ V''(x) = -\frac{6}{4} x = -\frac{3}{2} x \]

\[ V'''(20) = -30 < 0 \rightarrow x = +20 \]

is a local max.

1. This is the unique local extremum on \((0, \infty)\) so it is a global maximum there.

2. \[ V''(x) = -\frac{3}{2} x \] so \(V\) is concave down on \((0, \infty)\) so it has at most one absolute local maximum.

3. [Graph of a function with local maxima at \(-20\) and \(+20\)]
Question 6.
The manager of a 100-unit apartment complex knows from experience that all the units will be occupied if the is $800 per month. A market survey indicates that one additional unit will remain vacant for each $10 increase in rent. What rent should the manager charge to maximize revenue?

\[ p = Aq + B \]

\[ 0 \leq q \leq 100 \]

\[ 800 \leq p \leq ? \]

\[ 100 = A \cdot 800 + B \]

\[ \Delta q = A \Delta p \]

\[ A = \frac{\Delta q}{\Delta p} = \frac{-1}{10} \]

\[ 100 = -\frac{1}{10} \cdot 800 + B = -80 + B \]

\[ B = 180 \]

\[ q = \frac{-1}{10} p + 180 \]

\[ R(p) = p \cdot q = p \left( -\frac{1}{10} p + 180 \right) = -\frac{1}{10} p^2 + 180p \]

\[ \frac{dR}{dp} = -\frac{1}{5} p + 180 \]

\[ +\frac{1}{5} p = 180 \]

\[ p = 900 \]
Question 7.
You stand on a cliff at point \((0,0)\) overlooking a river. You see a boat due north at point \((0,2)\). The boat is traveling down the river along the curve \(y = \sqrt{x + 4}\) towards the harbour at \((-4,0)\). You want to wave to the boat at the point where it is closest to you. Find the coordinates of this point.

\[
d^2 = x^2 + y^2
\]

\[
y = \sqrt{x + 4}
\]

Target function:
\[
f(x) = d^2(x) = x^2 + y^2 = x^2 + x + 4
\]

\[
f'(x) = 2x + 1 = 0
\]

\[
x = -\frac{1}{2}
\]

\[
y = \sqrt{-\frac{1}{2} + 4} = \sqrt{\frac{7}{2}}
\]
Question 8.
A tutoring company is offering a workshop for the upcoming exam in December. Market research suggests that setting the price of the solution at $p$ in dollars will yield

$$q(p) = 20(18 - 2\sqrt{p})$$

students registering. What price should they set in order to maximize revenue?

$$R(p) = q(p)p = 20(18 - 2\sqrt{p})p$$

$$= 20 \cdot 18 \cdot p - 40p^{3/2}$$

$$\frac{dR}{dp} = 20 \cdot 18 - 40 \cdot \frac{3}{2} \sqrt{p}$$

$$\frac{dR}{dp} = 0$$

$$20 \cdot 18 = 40 \cdot \frac{3}{2} \sqrt{p}$$

$$18 = 3 \sqrt{p}$$

$$6 = \sqrt{p}$$

$$p = 36$$

$E = -1$

$$R(0) = 0$$

$$R(36) = 20 \cdot (18 - 2 \cdot 6) \cdot 36 > 0$$

$$R(81) = 0$$
Question 9. You are planning a city tour for a group of 100 tourists. If you can sell $x$ bus tour tickets, you can offer them for $30 - x/4$ each. If you can sell $y$ train tour tickets, you can offer them for $70 - y/2$ each. How many bus tickets, and how many train tickets should you sell to the tourists in order to maximize revenue (you can only sell one type of ticket to each passenger).

\[
R = (30 - \frac{x}{4})x + (70 - \frac{y}{2})y
\]

\[X + Y = 100\]

\[y = 100 - x\]

\[0 \leq x \leq 100\]
\[0 \leq y \leq 100\]

\[
R(X) = (30 - \frac{x}{4})x + (70 - \frac{100 - x}{2})(100 - x)
\]

\[= (30 - \frac{x}{4})x + (20 + \frac{x}{2})(100 - x)\]

\[= 30x - \frac{x^2}{4} + 2000 - 200x + 50x - \frac{x^2}{2} = -\frac{3}{4}x^2 + 60x + 2000\]

\[\text{Concave down, parabola}\]

\[\frac{dR}{dx} = -\frac{3}{2}x + 60 = 0\]

\[x = 40, \quad y = 60\]

We have an absolute max:
Question 10. A steel company, ABC steel, manufactures nuts and bolts. When x nuts are produced, they can be sold for $-3x + 500$ dollars each. When y bolts are produced, they can be sold for $-y + 300$ dollars each. Assume that nuts and bolts weigh 0.5kg each. How many nuts and how many bolts must be produced to maximize the revenue from 100kg of steel? Justify your answer.

\[
R = (500 - 3x)x + (300 - y)y
\]

\[
0.5x + 0.5y = 100 \quad \text{kg}
\]

\[
x + y = 200
\]

\[
y = 200 - x
\]

\[
R(x) = (500 - 3x)x + (300 - (200 - x))(200 - x)
\]

\[
= (500 - 3x)x + (100 + x)(200 - x)
\]

\[
= -3x^2 + 500x + 20000 - 200x - x^2 - 100x
\]

\[
= -4x^2 + 600x + 20000
\]

\[
R'(x) = -8x + 600 = 0
\]

\[
x = \frac{600}{8} = 7.5 \quad y = 12.5
\]
Problem 1.
Approximate $\sqrt{73}$ (as a simple or decimal fraction) without using a calculator.

$64 < 73 < 81$, \( f(x) = \sqrt{x} \), \( f'(x) = \frac{1}{2\sqrt{x}} \)

\[ \sqrt{64} = 8 = f(64) \]
\[ \sqrt{81} = 9 = f(81) \]

\[ f'(64) = \frac{1}{2\sqrt{64}} = \frac{1}{2 \cdot 8} = \frac{1}{16} \]
\[ f'(81) = \frac{1}{2\sqrt{81}} = \frac{1}{2 \cdot 9} = \frac{1}{18} \]

Tangent of \( f \) at \((64, 8)\):
\[ y = \frac{1}{16} (x - 64) + 8 \]
\[ = \frac{1}{16} x + 4 \]

\[ \sqrt{73} \approx f(73) \approx \frac{1}{16} (73 - 64) + 8 \]
\[ = \frac{13}{16} = 8.562 \]

Tangent of \( f \) at \((81, 9)\):
\[ y = \frac{1}{18} (x - 81) + 9 \]

\[ \sqrt{73} \approx f(73) \approx \frac{1}{18} (73 - 81) + 9 \]
\[ = \frac{22}{9} = 8.555... \]
\[
\sqrt{73} \approx 8.544...
\]

Concave down → Tangent above graph → linear approximation is an over-estimate.

Concave up → Tangent below graph → linear approximation is an under-estimate.
\( f(x) = x^2 \) \( \longrightarrow \) \( f(2) \approx ? \)

\( f(1) = 1 \)
\( f'(x) = 2x \)
\( f'(1) = 2 \)

Tangent of \( f \) at \( (1, 1) \) is
\[ y = 2(x-1) + 1 \]

\( y = f(2) \approx 2 \cdot (2-1) + 1 = 2 \cdot 1 + 1 = 3 \)

Under estimate because \( f(x) \) is concave up.
\[ \frac{137}{16} \text{ and } \frac{77}{9} \text{ are our linear approximations at } \sqrt{73}. \text{ Both are over-estimates because } f \text{ is concave down.} \\
\text{But geal an approximation are they?} \\
\text{Theorem (Mean Value Theorem - MVT):} \\
\text{If } f(x) \text{ is the tangent of } f(x) \text{ at } x=a \text{ then there exist } c \text{ between } x \text{ and } a \text{ s.t. } f(x) - l(x) = \frac{1}{2} f''(c) (x-a)^2 \\
\sqrt{73} - \frac{77}{9} = \frac{1}{2} f''(c) (73-81)^2 \\
\text{So: If } |f''(c)| \leq M \text{ for any } c \text{ between } x \text{ and } a \text{ then } \\
|f(x) - l(x)| \leq \frac{1}{2} \cdot M \cdot (x-a)^2 \\
\text{Worst case error.}
\[ f(x) = \sqrt{x}, \quad f'(x) = \frac{1}{2\sqrt{x}}, \quad f''(x) = -\frac{1}{4x^{3/2}} \]

Tangent of \( f \) at \( a = 64 \):
\[ \ell(x) = \frac{1}{16} (x - 64) + 8 \]

\[ \varepsilon = |f(x) - \ell(x)| = \left| f(73) - \ell(73) \right| \leq ? \]

\[ |f''(x)| = \frac{1}{4x^{3/2}} \]

We are looking for \( 0 < M \) s.t.
\[ |f''(x)| \leq M \]

for any \( 64 \leq x \leq 73 \)
\[ |f''(x)| \text{ is decreasing} \]

\[ |f''(x)| \leq |f''(64)| = \frac{1}{4 \cdot 8^3} \]

\[ E \leq \frac{1}{2} \cdot \frac{1}{4 \cdot 8^3} \cdot (73 - 64)^2 \]

\[ = \frac{1}{64} \cdot 8^2 = \frac{1}{64} \]
Problem 2.
Approximate \( \ln(2) \) (as a simple or decimal fraction) without using a calculator.

\[
f(x) = \ln(x) \quad f(2) \approx ?
\]

\[
1 < 2 < e
\]

\[
f(1) = 0 \quad f(e) = 1
\]

\[
f'(x) = \frac{1}{x} \quad f'(e) = \frac{1}{e}
\]

Tangent of \( f(x) \) at \((1, 0)\)

\[
l(x) = y = 1 \cdot (x-1) + 0 = x-1
\]

So

\[
l(2) = 2 - 1 = 1.
\]

\[
\ln(2) \approx 0.69...
\]
Linear approximations:

**Problem:**

1. Give an approximation of \( \ln(0.9) \)
2. Was this an overestimate or an underestimate?
3. Give a bound for the worst-case error in that approximation.

**Solution:**

1. \( f(x) = \ln(x) \)

   We wrote the tangent of \( f(x) \) at \((1,0)\):

   \[
   l(x) = x - 1
   \]

   \[
   \ln(0.9) = f(0.9) \approx l(0.9) = 0.9 - 1 = -0.1
   \]

2. \( f(x) \) is concave down on \((0.9, 1)\)
   and hence this is an overestimate.

\[
\begin{align*}
  f(1) &= 0 \\
  f'(1) &= 1 \\
  l(x) &= 1 \cdot (x-1) + 0 = x - 1
\end{align*}
\]
\[ E = f(0.9) - \ell(0.9) \]

\[ |E| \leq \frac{M}{2} (x-a)^2 \]

\[ a = 1 \]
\[ x = 0.9 \]

\[ f''(c) \leq M \text{ for } c \text{ between } 0.9 \text{ and } x. \]

\[ f(x) = \ln(x) \quad f'(x) = \frac{1}{x} \quad f''(x) = -\frac{1}{x^2} \]

\[ g(x) = |f''(x)| = \frac{1}{x^2} \]

\[ g(x) \text{ is decreasing on } [0.9, 1] \]

So we can take \( M = g(0.9) = \frac{1}{0.9^2} = \left(\frac{10}{9}\right)^2 \)

\[ |E| \leq \frac{\left(\frac{10}{9}\right)^2}{2} (0.9-1)^2 = \frac{\left(\frac{10}{9}\right)^2}{2} \cdot \frac{1}{16^2} = \frac{1}{16^2} = \frac{1}{256} \]
$F - 2012:$

The same question for $\cos(0.25)$.

Solution:

1. $f(x) = \cos(x)$
   $f'(x) = -\sin(x)$
   
   \[
   \begin{align*}
   f(0) &= 1 \\
   f'(0) &= 0
   \end{align*}
   \]
   So the tangent to $\cos(x)$ at $(0, 1)$ is $b(x) = 1$.

   In particular, $\cos(0.25) \approx b(0.25) = 1$.

2. $f(x)$ is concave down on $[0, 0.25]$ and hence it's an overestimate.

Alternatively: $-1 \leq \cos(x) \leq 1$

So $1$ must be an overestimate.
\( \frac{M}{2} (x-a)^2 \)

\( a = 0 \)

\( x = 0.25 \)

\( g(x) = |f''(x)| = |-\cos(x)| = |\cos(x)| \leq 1 \)

We take \( M = 1 \).

And \( |E| \leq \frac{1}{2} \cdot (0.25 - 0)^2 = \frac{1}{2} \cdot \left( \frac{1}{4} \right)^2 = \frac{1}{32} \cdot y^2 \)

\( M \geq |f'(c)| \)

\( P_{es} \in [a, b] \)

\( A \cdot x + B + \frac{M}{2} (x-c)^2 \)

\( A \cdot x + B \)

\( A \cdot x + B - \frac{M}{2} (x-c)^2 \)

\( \text{tangent} \)

\( a \)

\( b \)
Tayler polynomials:

\[\ln(2) \approx ?\]

\[f(x) = \ln(x) \quad f(1) = 0\]
\[f'(x) = \frac{1}{x} \quad f'(1) = 1\]
\[T_1(x) = 1(x - 1) + 0 = x - 1\]

\[0.69... = \ln(2) = f(2) \approx T_1(2) = 2 - 1 = 1\]

\[T_1(1) = f(1) \quad \leftarrow \text{value.}\]
\[T_1'(1) = f'(1) \quad \leftarrow \text{slope.}\]

---

Let's try to find

\[T_2(x) = A + B(x - 1) + C(x - 1)^2\]

\[T_1(x) = A + B(x - 1)\]

with:

\[T_2(1) = f(1)\]
\[T_2'(1) = f'(1)\]
\[T_2''(1) = f''(1)\]

\[A + Bx + Cx^2\]
\[R_3 + 2C_1x\]
\[2C\]
\[ \begin{align*}
\{ & P(1) = 0 \\
& P'(1) = 1 \\
& P''(1) = -1 \\
& f'(x) = -\frac{1}{x^2} = -x^{-2} \\
\end{align*} \]

\[ \begin{align*}
\{ & T_2(1) = A + B \cdot 0 + C \cdot 0 = A \\
& T_2'(1) = B \\
& T_2''(1) = 2C \\
\end{align*} \]

\[ A = 0 \]
\[ B = 1 \]
\[ 2C = -1 \quad C = -\frac{1}{2} \]

\[ T_2(x) = (x-1) - \frac{1}{2} (x-1)^2 \]

\[ 0.69... \approx \ln(2) \approx T_2(2) = \frac{1}{2} . \]
\[ T_3(x) = A + B(x-1) + C(x-1)^2 + D(x-1)^3 + E(x-1)^4 \]

\[
\begin{align*}
A &= T_3(1) = f(1) = 0 \\
B &= T_3'(1) = f'(1) = 1 \\
C &= T_3''(1) = f''(1) = -1 \\
D &= T_3'''(1) = f'''(1) = 2 \\
E &= f''''(x) = \frac{2}{x^3} 
\end{align*}
\]

\[
T_3'(x) = B + 2C(x-1) + 3D(x-1)^2 + 4E(x-1)^3
\]

\[
T_3''(x) = 2C + 6D(x-1) + 12E(x-1)^2
\]

\[
T_3'''(x) = 6D = 3 \cdot 2 \cdot 1 \cdot 0 + 24E \cdot (x-1)
\]

\[ A = 0 \]

\[ B = 1 \]

\[ C = -\frac{1}{2} \]

\[ D = \frac{1}{3} \]

\[ T_3(x) = (x-1) \cdot \frac{1}{2}(x-1)^2 + \frac{1}{3}(x-1)^3 \]

\[ 0.69 \approx l_n(2) \approx T_3(2) \approx \frac{5}{6} \approx 0.83 \]
Given a function \( f(x) \), the \( n \)th Taylor polynomial of \( f(x) \) at \( x = a \) is

\[
T_n(x) = a_0 + a_1(x-a) + a_2(x-a)^2 + \cdots + a_n(x-a)^n
\]

with:

\[
\begin{align*}
T_n(a) &= f(a) \\
T'_n(a) &= f'(a) \\
T''_n(a) &= f''(a) \\
& \vdots \\
T^{(n)}_n(a) &= f^{(n)}(a)
\end{align*}
\]

\[
\begin{align*}
a_0 &= f(a) \\
a_1 &= f'(a) \\
a_2 &= \frac{1}{2!} f''(a) \\
a_3 &= \frac{1}{3!} f'''(a) \\
& \vdots \\
a_n &= \frac{1}{n!} f^{(n)}(a)
\end{align*}
\]

\[
k! = 1 \cdot 2 \cdot 3 \cdots k
\]

0! = 1
1! = 1
2! = 2
3! = 6
4! = 24
5! = 120

\[
Q_k = \frac{1}{k!} f^{(k)}(a)
\]

\[
1! = 0! - 1
\]
Example: $e^x$ around $x=0$.

\[ f(x) = e^x \]
\[ f'(x) = e^x \]
\[ f''(x) = e^x \]
\[ \vdots \]
\[ f^{(n)}(x) = e^x \]

\[ f(0) = f'(0) = f''(0) = \ldots = f^{(n)}(0) = 1 \]

- $T_1(x) = 1 + (x-0) = 1 + x$
- $T_2(x) = 1 + x + \frac{1}{2}x^2$
- $T_3(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3$
- $T_4(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{6}x^3 + \frac{1}{24}x^4$

\[ f(1) = e \approx T_3(1) = 1 + 1 + \frac{1}{2} + \frac{1}{6} = 2 + \frac{1}{6} = 2.33 \]

If $a=0$ we call $T_n(x)$ the $n^{th}$ Maclaurin polynomial of $f(x)$.
Example: Find the 4th Maclaurin polynomial for $\cos(x), \sin(x)$.

- $f(x) = \cos(x)$
  - $f(0) = 1$
- $f'(x) = -\sin(x)$
  - $f'(0) = 0$
- $f''(x) = -\cos(x)$
  - $f''(0) = -1$
- $f'''(x) = \sin(x)$
  - $f'''(0) = 0$
- $f^{(4)}(x) = \cos(x)$
  - $f^{(4)}(0) = 1$

The 4th Maclaurin polynomial is:

$$T_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24}$$

- $f(x) = \sin(x)$
  - $f(0) = 0$
- $f'(x) = \cos(x)$
  - $f'(0) = 1$
- $f''(x) = -\sin(x)$
  - $f''(0) = 0$
- $f'''(x) = -\cos(x)$
  - $f'''(0) = -1$
- $f^{(4)}(x) = \sin(x)$
  - $f^{(4)}(0) = 0$

The 4th Maclaurin polynomial is:

$$T_4(x) = x - \frac{x^3}{6}$$
Example: Find 4th Maclaurin poly. of 

\[ f(x) = e^x \cos(x) \quad \text{at } a = 0 \]

\[ f'(x) = e^x \cos(x) - e^x \sin(x) \]

\[ f''(x) = (e^x \cos(x) - e^x \sin(x)) - (e^x \sin(x) + e^x \cos(x)) = -2e^x \sin(x) \]

\[ f'''(x) = -2(e^x \sin(x) + e^x \cos(x)) \]

\[ f^{(4)}(x) = -2(e^x \sin(x) + e^x \cos(x)) - 2(e^x \cos(x) - e^x \sin(x)) = -4e^x \cos(x) \]

\[ f(0) = 1, \quad f'(0) = 1, \quad f''(0) = 0, \quad f^{(3)}(0) = -2, \quad f^{(4)}(0) = 4 \]

\[ T_4(x) = 1 + 1 \cdot x + \frac{1 \cdot 2}{2} x^2 + \frac{(-2) \cdot 3}{6} x^3 + \frac{(-4) \cdot 4}{24} x^4 \]

\[ = 1 + x - \frac{1}{3} x^3 - \frac{1}{6} x^4 \]
$e^x$: $P_4(x) = 1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24} + \cdots$

$\cos(x)$: $Q_4(x) = 1 - \frac{x^2}{2} + \frac{x^4}{24} + \cdots$

$p(x) = e^x \cos(x)$

$P_4(x) \cdot Q_4(x) = (1 + x + \frac{x^2}{2} + \frac{x^3}{6} + \frac{x^4}{24})(1 - \frac{x^2}{2} + \frac{x^4}{24}) + \cdots$

$\neq \frac{1}{2} + x - \frac{x^2}{2} + \frac{x^3}{2} - \frac{x^3}{2} + \frac{x^4}{24} - \frac{x^4}{4} + \frac{x^4}{24}$

$= 1 + x - \frac{3}{6}x^3 + (\frac{1}{24} - \frac{1}{4} + \frac{1}{24})x^4 + \cdots$

$= 1 + x - \frac{1}{3}x^3 - \frac{1}{6}x^4 = T_4(x)$