1. (15 marks)

(a) (3 marks) Let

\[ f(x, y) = y^2 + y \ln x. \]

Compute both first-order partial derivatives of \( f \) at the point \((1, 2)\). Simplify your answers.

Solution: Calculate the first-order partial derivatives of \( f \):

\[ f_x(x, y) = \frac{y}{x}, \quad f_y(x, y) = 2y + \ln x. \]

Evaluate \( f_x \) and \( f_y \) at \((1, 2)\):

\[ f_x(1, 2) = 2/1 = 2, \quad f_y(1, 2) = 2(2) + \ln 1 = 4. \]

Thus, \( f_x(1, 2) = 2 \), and \( f_y(1, 2) = 4 \).

(b) (2 marks) Given \( f(x, y) \) as in part (a), sketch the trace of the surface \( z = f(x, y) \) in the \( x = 1 \) plane.

Solution: In the \( x = 1 \) plane, the trace of the surface is given by the equation \( z = y^2 + y \ln 1 = y^2 \), which is a parabola in the \( yz \)-plane.

(c) (2 marks) Find a unit vector parallel to \( \langle -2, 1, 2 \rangle \).

Solution: Let \( \mathbf{v} \) be such a vector. Then, since \( \mathbf{v} \) is parallel to \( \langle -2, 1, 2 \rangle \), we have that \( \mathbf{v} = c\langle -2, 1, 2 \rangle = \langle -2c, 1, 2c \rangle \) for some real constant \( c \). Since \( \mathbf{v} \) is a unit vector, which means it has length 1, we get:

\[ 1 = \sqrt{(-2c)^2 + c^2 + (2c)^2} = \sqrt{9c^2} = 3|c| \]
\[ 1/3 = |c| \]
\[ c = \pm 1/3. \]

Thus, a unit vector in \( \mathbb{R}^3 \) which is parallel to the vector \( \langle -2, 1, 2 \rangle \) can either be \( \langle -2/3, 1/3, 2/3 \rangle \) and \( \langle 2/3, -1/3, -2/3 \rangle \).
(d) (2 marks) Find an equation for the plane passing through the point $P(1, 2, 3)$ that is orthogonal to the vector $\langle 4, 0, -1 \rangle$.

**Solution:** An equation of the plane passing through the point $P(1, 2, 3)$ that is orthogonal to the vector $\langle 4, 0, -1 \rangle$ is:

$$4(x - 1) + 0(y - 2) - 1(z - 3) = 0$$

$$\Rightarrow 4x - z = 1.$$

(e) (3 marks) Determine if the plane described by the equation:

$$2x - 5y + 2z = -1,$$

is orthogonal to the plane given in part (d).

**Solution:** To determine if the plane $2x - 5y + 2z = -1$ is orthogonal to the plane obtained in part (d), we want to determine if their normal vectors are orthogonal to each other, that is, whether $\langle 2, -5, 2 \rangle$ is orthogonal to $\langle 4, 0, -1 \rangle$.

Since:

$$2(4) + (-5)(0) + 2(-1) = 6 \neq 0,$$

$\langle 2, -5, 2 \rangle$ is not orthogonal to $\langle 4, 0, -1 \rangle$. Thus, the planes are not orthogonal to each other.

(f) (3 marks) Assume that $f(x, y)$ has continuous partial derivatives of all orders. If $f_y(x, y) = x^3 + 2x^2y$, compute $f_{xyx}$. State in detail any result that you use.

**Solution:** Since $f(x, y)$ has continuous partial derivatives of all orders, in particular, $f_{xy}$ and $f_{yx}$ are continuous. By Clairaut’s Theorem, that means $f_{xy} = f_{yx}$. Thus, $f_{xyx} = (f_{xy})_x = (f_{yx})_x = f_{yxx}$. Hence, to find $f_{xyx}$, we have:

$$f_{yx}(x, y) = 3x^2 + 4xy,$$

$$f_{yxx} = 6x + 4y.$$

Thus, $f_{xyx} = 6x + 4y$.

2. (5 marks) Consider the surface $S$ given by:

$$z^2 = x - 9y^2.$$
(a) (4 marks) Find and sketch the level curves of \( S \) for \( z_0 = 1 \) and \( z_0 = 2 \).

**Solution:** For \( z_0 = 1 \), the level curve is \( 1 = x - 9y^2 \), which is, \( x = 9y^2 + 1 \) (a parabola in the \( xy \)-plane).

For \( z_0 = 2 \), the level curve is \( 4 = x - 9y^2 \), which is, \( x = 9y^2 + 4 \) (a parabola in the \( xy \)-plane).

(b) (1 mark) Based on the traces you sketched above, which of the following renderings represents the graph of the surface?

**Solution:** The answer is (A), since in (B), the level curves at \( z_0 = 1 \) and \( z_0 = 2 \) seem to be empty, not parabolas.

3. (10 marks) Let \( R \) be the semicircular region \( \{x^2 + y^2 \leq 4, x \geq 0\} \). Find the maximum and minimum values of the function

\[
f(x, y) = x^2 - 2x + y^2.
\]
on the boundary of the region \( R \).

**Solution:** The boundary of the region \( R \) consists of two pieces: the semicircular arc which can be parametrized by \( x = 2 \cos \theta \) and \( y = 2 \sin \theta \) for \( -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \), and the vertical segment \( x = 0 \) for \(-2 \leq y \leq 2\). We will find the potential candidates where the maximum and minimum can occur on each piece:

- **On the semicircular arc:** We have that \( f(x,y) = g(\theta) = (2 \cos \theta)^2 + (2 \sin \theta)^2 - 2 \) for \(-\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2} \). Then, \( g'(\theta) = 4 \sin \theta = 0 \) if and only if \( \theta = 0 \). So, there are 3 points where extrema can occur: \((2,0)\) (critical point), \((0,-2)\) and \((0,2)\) (end points).

- **On the vertical segment:** We have that \( f(0,y) = h(y) = y^2 \) for \(-2 \leq y \leq 2\). So, \( h'(y) = 2y = 0 \) when \( y = 0 \). So, there are 3 points where extrema can occur: \((0,0)\) (critical point), \((0,-2)\) and \((0,2)\) (end points).

Evaluate \( f \) at those points, we get:

\[
\begin{align*}
    f(2,0) &= 0, \\
    f(0,-2) &= 4, \\
    f(0,2) &= 4, \\
    f(0,0) &= 0
\end{align*}
\]

Thus, on the boundary of \( R \), \( f \) attains the absolute maximum value 4 at the points \((0,-2)\) and \((0,2)\) and the absolute minimum value 0 at the points \((2,0)\) and \((0,0)\).

4. (10 marks) Find all critical points of the following function:

\[
f(x,y) = xy - \frac{x^2}{2} - \frac{y^3}{3} + 5.
\]

Classify each point as a local minimum, local maximum, or saddle point.

**Solution:** Compute the first-order partial derivatives of \( f \):

\[
\begin{align*}
    f_x(x,y) &= y - x \\
    f_y(x,y) &= x - y^2.
\end{align*}
\]

Since both \( f_x \) and \( f_y \) are defined at every point in \( \mathbb{R}^2 \), the only critical points of \( f \) are those at which \( f_x = f_y = 0 \). If \( f_x = 0 \), then \( x = y \). Replacing \( x = y \) into \( f_y = 0 \), we get:

\[
y - y^2 = y(1-y) = 0 \Rightarrow y = 0, 1.
\]

So, we get two critical points \((0,0)\) and \((1,1)\). Compute the second-order partial derivatives and the discriminants,

\[
\begin{align*}
    f_{xx} &= -1, \\
    f_{yy} &= -2y, \\
    f_{xy} &= 1, \\
    D(x,y) &= 2y - 1
\end{align*}
\]

Using the Second Derivative Test to classify the points, we get:
• At the point (0, 0), \( D(0, 0) = -1 < 0 \), so \( (0, 0) \) is a saddle point.

• At the point (1, 1), \( D(1, 1) = 1 > 0 \) and \( f_{xx}(1, 1) = -1 < 0 \), so \( (1, 1) \) is a local maximum.

5. (10 marks) A firm produces:

\[ P(x, y) = x^{\frac{2}{3}}y^{\frac{1}{3}} \]

units of goods per week, utilizing \( x \) units of labour and \( y \) units of capital. If labour costs $27 per unit, and capital costs $0.5 per unit, use the method of Lagrange multiplier to find the most cost-efficient division of labour and capital that the firm can adopt if its goal is to produce 6 units of goods per week.

Clearly state the objective function and the constraint. You are not required to justify that the solution you obtained is the absolute maximum. A solution that does not use the method of Lagrange multipliers will receive no credit, even if it is correct.

**Solution:** Since labour costs $27 per unit, and capital costs $0.5 per unit, the cost function is \( C(x, y) = 27x + 0.5y \). The objective function to minimize is the cost function \( C(x, y) = 27x + 0.5y \), and the constraint function is \( g(x, y) = x^{\frac{2}{3}}y^{\frac{1}{3}} - 6 = 0 \) (since its goal is to produce 6 units). Using Lagrange multiplier, we need to solve the following system of equations:

\[
\nabla C(x, y) = \lambda \nabla g(x, y)
\]

\[
g(x, y) = 0
\]

More explicitly, we need to solve:

\[
27 = \frac{2}{3} \lambda x^{\frac{1}{3}} y^{\frac{1}{3}}
\]

\[
0.5 = \frac{1}{3} \lambda x^{\frac{2}{3}} y^{\frac{2}{3}}
\]

\[
x^{\frac{2}{3}}y^{\frac{1}{3}} - 6 = 0.
\]

Isolate for \( \lambda \) in the first two equations, we get:

\[
\lambda = \frac{81}{2} x^{\frac{1}{3}} y^{\frac{1}{3}}, \quad \lambda = \frac{3}{2} x^{\frac{2}{3}} y^{\frac{2}{3}}.
\]

Equate the above equations, we get:

\[
\frac{81}{2} x^{\frac{1}{3}} y^{\frac{1}{3}} = \frac{3}{2} x^{\frac{2}{3}} y^{\frac{2}{3}}
\]

\[
27x = y.
\]
Replace $27x = y$ in the third equation, we get $3x - 6 = 0$, so $x = 2$. Thus, $y = 54$ and $\lambda = 27/2$. Therefore, if the firm uses 2 units of labour and 54 units of capital, then it can produce 6 units of good while minimizing its costs.