THE WEYL CHARACTER FORMULA

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We prove the Weyl character formula in this note. All notations will be the same as in the note on the theorem of highest weight. Let $\mathfrak{g}$ be a finite dimensional complex semisimple Lie algebra, $\mathfrak{h}$ a Cartan subalgebra of $\mathfrak{g}$, Φ the set of roots of $\mathfrak{g}$ wrt $\mathfrak{h}$, Δ a basis for Φ, Φ^+ and Φ^- the subsets of positive, resp. negative roots wrt Δ, $\mathcal{W}$ the Weyl group and ρ the half-sum of the positive roots.

1 The Weyl Character Formula

Let $\mathbb{Z}[\mathfrak{h}^*]$ be the group ring of the Abelian group $\mathfrak{h}^*$, i.e. for each $\mu \in \mathfrak{h}^*$, there is a basis element $e^\mu \in \mathfrak{h}^*$ with multiplication given by $e^\mu e^\nu = e^{\mu+\nu}$ for $\mu, \nu \in \mathfrak{h}^*$. $\mathbb{Z}[\mathfrak{h}^*]$ is an integral domain. On $\mathbb{Z}[\mathfrak{h}^*_\mathbb{R}]$, the Weyl group acts naturally: for $w \in \mathcal{W}$ and $\mu \in \mathfrak{h}^*_\mathbb{R}$ we have $w \cdot e^\mu = e^{w\mu}$.

Definition 1.1. For a finite dimensional $\mathfrak{g}$-module V, the character of V, $\mathrm{ch}V \in \mathbb{Z}[\mathfrak{h}^*]$ is defined as

$$\mathrm{ch}V = \sum_{\mu \in \mathfrak{h}^*} (\mathrm{dim}V_\mu) e^\mu$$

When λ is a dominant integral weight, the module $L(\lambda)$ is finite dimensional. The Weyl character formula gives an expression for the character of $L(\lambda)$.

Theorem 1.2. (The Weyl Character Formula) For λ a dominant integral weight,

$$\mathrm{ch}L(\lambda) = \frac{\sum_{w \in \mathcal{W}} \det(w) e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

If $\lambda = 0$, $L(0)$ is the trivial representation of L, so $\mathrm{ch}L(0) = e^0$ and the character formula gives the Weyl denominator formula,

$$\sum_{w \in \mathcal{W}} (\mathrm{det}w) e^{w(\rho)} = \prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})$$

So the character formula can be written as

$$\mathrm{ch}L(\lambda) = \frac{\sum_{w \in \mathcal{W}} \det(w) e^{w(\lambda+\rho)}}{\sum_{w \in \mathcal{W}} \det(w) e^{w(\rho)}}$$

This form of the character formula is useful in computing the dimension of $L(\lambda)$. We state the theorem here, a proof can be found in the references.

Theorem 1.3. (The Weyl Dimension Formula)

$$\mathrm{dim}L(\lambda) = \frac{\prod_{\alpha \in \Phi^+} (\lambda + \rho, \alpha)}{\prod_{\alpha \in \Phi^+} (\rho, \alpha)}$$

In the following lemma, the computations are formal, but make sense. We write down the 'character of the Verma modules' (this is a formal object, really, as the Verma modules are infinite dimensional).

Lemma 1.4.

$$\mathrm{ch}\Delta(\lambda) = \frac{e^\lambda}{\prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})} = \frac{e^{\lambda+\rho}}{\prod_{\alpha \in \Phi^+} (e^{\alpha/2} - e^{-\alpha/2})}$$

Proof. Consider the product: $e^\lambda \cdot \prod_{\alpha \in \Phi^+} (1 + e^{-\alpha} + e^{-2\alpha} + \cdots)$. For ν positive, the coefficient of $e^{\lambda-\nu}$ in this product is equal to the number of ways to write ν as an (unordered) sum of positive roots with non-negative integer coefficients. Recalling that $\Delta(\lambda)$ is a free $U(\mathfrak{n})$-module of rank 1, this number is precisely the dimension of $\Delta(\lambda)_{\lambda-\nu}$. Hence the product is $\mathrm{ch}\Delta(\lambda)$ and the equalities follow from the computation: $1 + e^{-\alpha} + e^{-2\alpha} + \cdots = \frac{1}{1-e^{-\alpha}}$. □

We record two identities for later use.

Lemma 1.5. For λ a dominant integral weight and $w \in \mathcal{W}$ the following two identities hold:

$$w \cdot \mathrm{ch}L(\lambda) = \mathrm{ch}L(\lambda)$$

$$w \cdot \left(e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \right) = \det w \left(e^{\rho} \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha}) \right)$$

Proof. To see that these hold, it is enough to prove both in the case $w = \sigma_\alpha$, a reflection corresponding to a simple positive root $\alpha \in \Delta$. The first follows from the fact that $\mathrm{dim}L(\lambda)_\mu = \mathrm{dim}L(\lambda)_{\sigma_\alpha \mu}$ (proved earlier) and the second folows from the fact that σ_α permutes the positive roots other than α so that $\sigma_\alpha(\rho) = \rho - \alpha$. □

2 Some properties of the Verma modules

Recall that the Casimir operator $\mathcal{C}$ in $U(\mathfrak{g})$ may be defined as follows: Let κ be the Killing form of $\mathfrak{g}$. For each positive root α, let x_α, y_α be chosen from $\mathfrak{g}_\alpha, \mathfrak{g}_{-\alpha}$ respectively such that $\kappa(x_\alpha, y_\alpha) = 1$ and let $\{h_1, h_2, \cdots, h_r\}$ be an orthonormal basis of $\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ ($\kappa|_{\mathfrak{h}\times\mathfrak{h}}$ is a nondegenerate bilinear form on $\mathfrak{h}$). Then

$$\mathcal{C} = \sum_{\alpha \in \Phi^+} (x_\alpha y_\alpha + y_\alpha x_\alpha) + \sum_{i=1}^{r} h_i^2$$

Proposition 2.1. For $\lambda \in \mathfrak{h}^*$, $\mathcal{C}$ acts on $\Delta(\lambda)$ by the scalar $(\lambda + \rho, \lambda + \rho) - (\rho, \rho)$.

Proof. It is enough to know how $\mathcal{C}$ acts on the generator, v_λ. This is a computation:

$$\mathcal{C} v_\lambda = \left(\sum_{\alpha \in \Phi^+} (2 y_\alpha x_\alpha + [x_\alpha, y_\alpha]) + \sum_{i=1}^{r} h_i^2 \right) v_\lambda = \sum_{\alpha \in \Phi^+} \lambda([x_\alpha, y_\alpha]) v_\lambda + \sum_{i=1}^{r} \lambda(h_i)^2 v_\lambda$$

So the scalar by which $\mathcal{C}$ operates is $\sum_{\alpha\in\Phi^+} \lambda([x_\alpha, y_\alpha]) + \sum_{i=1}^{r} \lambda(h_i)^2$. Let t_λ be the element of $\mathfrak{h}$ such that $\lambda(\cdot) = \kappa(t_\lambda, \cdot)$. Then

$$\begin{aligned} \sum_{\alpha \in \Phi^+} \lambda([x_\alpha, y_\alpha]) &= \sum_{\alpha \in \Phi^+} \kappa(t_\lambda, [x_\alpha, y_\alpha]) = \sum_{\alpha \in \Phi^+} \kappa([t_\lambda, x_\alpha], y_\alpha) \\ &= \sum_{\alpha \in \Phi^+} \kappa(\alpha(t_\lambda) x_\alpha, y_\alpha) = \sum_{\alpha \in \Phi^+} \alpha(t_\lambda) = 2\rho(t_\lambda) = 2(\rho, \lambda) \end{aligned}$$

and

$$\sum_{i=1}^{r} \lambda(h_i)^2 = \sum_{i=1}^{r} \kappa(t_\lambda, h_i)^2 = \sum_{i=1}^{r} \kappa(t_\lambda, \kappa(t_\lambda, h_i) h_i) = \kappa(t_\lambda, t_\lambda) = (\lambda, \lambda)$$

So the scalar is $2(\rho, \lambda) + (\lambda, \lambda) = (\lambda + \rho, \lambda + \rho) - (\rho, \rho)$. □

Theorem 2.2. For $\lambda \in \mathfrak{h}^*_{\mathbb{R}}$, $\Delta(\lambda)$ has a finite filtration

$$\Delta(\lambda) = M_0 \supset M_1 \supset \cdots \supset M_{n-1} \supset M_n = (0)$$

such that the succesive quotients M_{i-1}/M_i are isomorphic to $L(\mu_i)$, and the μ_i belong to the set $S_\lambda = \{\mu \in \mathfrak{h}^*_{\mathbb{R}} \mid \mu \leq \lambda, (\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)\}$.

Proof. First note that if M/N is a simple subquotient of $\Delta(\lambda)$, then $M/N \cong L(\mu)$ for some $\mu \in S_\lambda$. Indeed, the set of weights of M/N is a subset of the weights of $\Delta(\lambda)$. Hence M/N has a maximal weight, μ implying that $M/N \cong L(\mu)$ (see the note on the theorem of highest weight). From 2.1, $\mathcal{C}$ acts on $\Delta(\lambda)$ by $(\lambda+\rho, \lambda+\rho)$. So it also acts on any subquotient of $\Delta(\lambda)$ by the scalar $(\lambda + \rho, \lambda + \rho)$. But as $M/N \cong L(\mu)$, $\mathcal{C}$ acts on M/N by $(\mu + \rho, \mu + \rho)$. This shows $(\mu + \rho, \mu + \rho) = (\lambda + \rho, \lambda + \rho)$ and $\mu \in S_\lambda$.

In S_λ, the condition $\mu \leq \lambda$ implies that μ lies in a discrete set and the equality $(\mu+\rho, \mu+\rho) = (\lambda+\rho, \lambda+\rho)$ implies that μ lies in the compact set which is a translate of the sphere of radius $\sqrt{(\lambda + \rho, \lambda + \rho)}$ by $-\rho$. So S_λ is contained in the intersection of a discrete set with a compact set and so is finite.

We construct the filtration: $M_1 = \mathrm{rad}(\Delta(\lambda))$ and if M_i has been constructed and is nonzero, let M_{i+1} be a maximal submodule of M_i. We claim that the chain $M_0 \supset M_1 \supset \cdots$ ends at (0) for some i. This is because only the finitely many $\mu \in S_\lambda$ can occur as the highest weights of the subquotients M_{i-1}/M_i and each such μ can occur only finitely many times as the highest weight because the weight spaces $\Delta(\lambda)_\mu$ are finite dimensional. This proves the theorem. □

Remark: The theorem holds for all $\lambda \in \mathfrak{h}^*$. I am not sure if the proof above works for the general case as $(\cdot, \cdot)$ is an inner product on $\mathfrak{h}^*_{\mathbb{R}}$ only, so the set S_λ may not be compact. Infact, one proves that the BGG category $\mathcal{O}$ (of which the Verma modules are objects and the simple objects are precisely the $L(\mu)$) is Artinian by Harish-Chandra's theorem on characters and that's the only proof I've seen.

3 Proof of the Weyl character formula

Proof. The idea is to use Theorem 2.2 and an 'inversion' trick to express $\text{ch}L(\lambda)$ in terms of $\text{ch}\Delta(\mu)$, $\mu \in S_\lambda$, which we know well. For $\mu, \nu \in S_\lambda$, let $[\Delta(\nu) : L(\mu)]$ be the number of simple subquotients isomorphic to $L(\mu)$ in a composition series for $\Delta(\nu)$ (In view of Theorem 2.2 and the Jordan-Holder theorem, this number is well defined). Since the character is additive,

$$\text{ch}\Delta(\nu) = \sum_{\mu \in S_\nu} [\Delta(\nu) : L(\mu)]\text{ch}L(\mu)$$

and $[\Delta(\nu) : L(\nu)] = 1$ because the λ-weight space of $\Delta(\lambda)$ is one dimensional. If $\nu \in S_\lambda$, then $S_\nu \subset S_\lambda$, and hence in an appropriate ordering of the ν's, the coefficients $[\Delta(\nu) : L(\mu)]$ form an upper triangular matrix with non-negative integer entries and 1s on the diagonal. We can invert this matrix to get

$$\text{ch}L(\lambda) = \sum_{\mu \in S_\lambda} c(\mu, \lambda)\text{ch}\Delta(\mu) \tag{1}$$

for some coefficients $c(\mu, \lambda) \in \mathbb{Z}$ and $c(\lambda, \lambda) = 1$. It remains to compute these unknown coefficients. Multiplying both sides by e^ρ and using Lemma 1.3, we get

$$\text{ch}L(\lambda)\left(e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})\right) = \sum_{\mu \in S_\lambda} c(\mu, \lambda)e^{\mu+\rho}$$

Applying $w \in \mathcal{W}$ to both sides and using Lemma 1.4,

$$\det(w)\text{ch}L(\lambda)\left(e^\rho \prod_{\alpha \in \Phi^+} (1 - e^{-\alpha})\right) = \sum_{\mu \in S_\lambda} c(\mu, \lambda)e^{w(\mu+\rho)}$$

In particular, the following equation holds for all $w \in \mathcal{W}$:

$$\sum_{\mu \in S_\lambda} \det(w)c(\mu, \lambda)e^{\mu+\rho} = \sum_{\mu \in S_\lambda} c(\mu, \lambda)e^{w(\mu+\rho)} \tag{2}$$

So let μ be such that $c(\mu, \lambda) \neq 0$. We claim that $\mu = w_\mu(\lambda + \rho) - \rho$ for a unique $w_\mu \in \mathcal{W}$ (The action $w \cdot \lambda := w(\lambda + \rho) - \rho$ is called the dot action of $\mathcal{W}$ on $\mathfrak{h}^*_\mathbb{R}$). We have seen that there exists a $w \in \mathcal{W}$ such that $w(\mu + \rho)$ is dominant. If $\nu = w \cdot \mu$, comparing the coefficients of $e^{\nu+\rho}$ in (2) gives $c(\mu, \lambda) = \det(w)c(\nu, \lambda)$; in particular, $\nu \in S_\lambda$ as $c(\nu, \lambda) \neq 0$. Then we get,

$$0 = (\lambda + \rho, \lambda + \rho) - (\nu + \rho, \nu + \rho) = (\lambda + \nu + 2\rho, \lambda - \nu) \geq 0$$

as $\lambda + \nu + 2\rho$ is strongly dominant and $\nu \leq \lambda$. The equality forces $\nu = \lambda$ hence, $w_\mu = w^{-1}$ works. To see uniqueness, if w_1, w_2 are such that $\mu = w_1 \cdot \lambda = w_2 \cdot \lambda$, then $w(\lambda + \rho) = \lambda + \rho$ for $w = w_1^{-1}w_2$. If w is not the identity, w sends some positive root α to a negative root, but in that case

$$0 < (\lambda + \rho, \alpha) = (w(\lambda + \rho), w(\alpha)) = (\lambda + \rho, w(\alpha)) < 0$$

is a contradiction. So $w_1 = w_2$ and we get: $c(\mu, \lambda) = \det(w_\mu)c(\lambda, \lambda) = \det(w_\mu)$. Conversely, it is obvious by the above that $\mu = w \cdot \lambda$ implies $c(\mu, \lambda) = \det(w)$. Using all of this information, equation (1) becomes:

$$\text{ch}L(\lambda) = \frac{\sum_{\mu \in S_\lambda} c(\mu, \lambda)e^{\mu+\rho}}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})} = \frac{\sum_{w \in \mathcal{W}} \det(w)e^{w(\lambda+\rho)}}{\prod_{\alpha \in \Phi^+}(e^{\alpha/2} - e^{-\alpha/2})}$$

and this completes the proof. □

References

[1] James E. Humphreys, *Representations of Semisimple Lie Algebras in the BGG Category O*, American Mathematical Society, 2008.

[2] Akhil Mathew, *Climbing Mount Bourbaki*, `https://amathew.wordpress.com/tag/semisimple-lie-algebras/`.