Deformation Theory

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This is a quick introduction to deformation theory. More generally, just a handy document to remember all the definitions and some motivation for them, with lesser emphasis on the proofs. Unless otherwise specified, we work over the field of complex numbers.

1 Introduction

The goal of algebraic geometry is to classify all varieties up to isomorphism. This is too hard to do but we'd at least like to parametrize our varieties of interest by a space, in this case, a scheme of finite type. The problem of classifying objects of a certain type is called a moduli problem. The first example to think of is probably the case of nonsingular curves of genus g.

For a scheme S of finite type, consider 'families' of genus g curves over S. A family of nonsingular genus g curves over S is given by the data of a proper, smooth morphism $f : \mathcal{X} \to S$ such that every geometric fiber is a nonsingular genus g curve. Two families $f : \mathcal{X} \to S$ and $f' : \mathcal{X}' \to S$ are isomorphic if there is an isomorphism $\phi : \mathcal{X} \to \mathcal{X}'$ over S, i.e. fitting into a commutative diagram

$$\begin{tikzcd} \mathcal{X} \arrow[rr, "\phi"] \arrow[dr, "f"'] & & \mathcal{X}' \arrow[dl, "f'"] \\ & S & \end{tikzcd}$$

Let $\mathbf{Sch_f}$ be the category of schemes of finite type. To formalize, define the functor $\mathcal{M}_g : \mathbf{Sch_f}^{op} \to \mathbf{Set}$

$$\mathcal{M}_g(S) := \{f : \mathcal{X} \to S \text{ is a family of nonsingular genus g curves}\}/\sim$$

where the equivalence $\sim$ is given by isomorphism of families. The functoriality is by pullbacks: If $h : T \to S$ is a morphism, then $F(h) : F(S) \to F(T)$ sends $f : \mathcal{X} \to S$ to the pullback family $f_T : \mathcal{X} \times_S T \to T$. Observe that $\mathcal{M}_g(\mathbb{C}) = \mathcal{M}_g(\mathrm{Spec}\mathbb{C})$ gives the objects that we're interested in. $\mathcal{M}_g$ is an example of a moduli functor.

Here are some more examples of moduli problems and moduli functors:

- Line bundles over a scheme: Let X be a fixed scheme. A family of line bundles parametrized by a scheme T is a line bundle $\mathcal{L}$ on $X \times_{\mathbb{C}} T$. Define the Picard functor, $\mathrm{Pic}_X : \mathbf{Sch_f}^{op} \to \mathbf{Set}$ by

$$\mathrm{Pic}_X(T) := \mathrm{Pic}(X \times_{\mathbb{C}} T)/p^*\mathrm{Pic}(T)$$

  where $p : X \times_{\mathbb{C}} T \to T$ is the projection. Here Pic is the ordinary Picard group. The functoriality is by pullbacks as before and $\mathrm{Pic}_X(\mathbb{C}) = \mathrm{Pic}(X)$, the objects of interest.

- Genus 0 curves with n marked points: Fix an integer $n \geq 1$. A family of nonsingular genus 0 curves with n marked points over $S \in \mathbf{Sch_f}$ is the data $\pi : \mathcal{X} \to S$ a smooth, proper map such that every geometric fiber is a nonsingular genus 0 curve along with n sections $s_i : S \to \mathcal{X}, i = 1, 2, \dots, n$ that do not pairwise intersect. An isomorphism of two families over S given by data $(\mathcal{X}, S, \pi, \{s_i\}), (\mathcal{X}', S, \pi', \{s_i'\})$ is an S-isomorphism of schemes $\phi : \mathcal{X} \to \mathcal{X}'$ such that the diagram commutes for every i.

$$\begin{tikzcd}
\mathcal{X} \arrow[rr, "\phi"] & & \mathcal{X}' \\
& S \arrow[ul, "s_i"] \arrow[ur, "s_i'"'] &
\end{tikzcd}$$

  Define the functor $\mathcal{M}_{0.n} : \mathbf{Sch_f}^{op} \to \mathbf{Set}$ by

$$\mathcal{M}_{0.n}(S) = \{(\mathcal{X}, S, \pi, \{s_i\}) \text{ a family of smooth genus 0 curves with } n \text{ marked points}\}/\sim$$

  where $\sim$ is again isomorphism of families. The functoriality is by pullbacks: the fiber product $(T \times_S \mathcal{X}) \times_{\mathcal{X}} S$ is T and the section s_i necessarily pulls back to a section t_i.

$$\begin{tikzcd}
T \arrow[r] \arrow[d, "t_i"'] & S \arrow[d, "s_i"] \\
T \times_S \mathcal{X} \arrow[r] \arrow[d, "\pi_T"] & \mathcal{X} \arrow[d, "\pi"] \\
T \arrow[r, "f"'] & S
\end{tikzcd}$$

- Let X be a projective scheme with a fixed very ample line bundle $\mathcal{O}_X(1)$. The Hilbert scheme functor $\mathbf{Hilb_X} : \mathbf{Sch_f}^{op} \to \mathbf{Set}$ of X is defined by: $\mathbf{Hilb_X}(S)$ to be the set of all closed subschemes $Z \subset X \times_{\mathbb{C}} S$, flat over S. For any numerical polynomial p, there is a subfunctor $\mathbf{Hilb}^p_\mathbf{X}$ of $\mathbf{Hilb_X}$ given by

$$\mathbf{Hilb}^p_\mathbf{X}(S) = \{\text{closed subschemes } Z \subset X \times S \text{ flat over } S, \text{ with } p(Z_s) = p\}$$

where $p(Z_s)$ denotes the Hilbert polynomial of Z_s by the induced polarization.

The hope is that these moduli functors are representable; if this happens our objects of interest and even families, can be obtained in the best way possible. A moduli functor $F : \mathbf{Sch_f}^{op} \to \mathbf{Set}$ is representable if there is an object $M \in \mathbf{Sch_f}$ and an isomorphism $\Phi : h_M = \mathrm{Hom}(\cdot, M) \to F$. This object is unique up to a unique isomorphism. Giving such an isomorphism Φ is equivalent, by the Yoneda lemma, to give an object of $F(M)$ corresponding to the identity morphism $1_M \in \mathrm{Hom}(M, M)$. Call this family $u : U \to M$. This family has extra properties:

1. Given a family $f : \mathcal{X} \to S$, there is a unique morphism $j_f : S \to M$ such that $f : \mathcal{X} \to S$ is isomorphic to the pullback family $u_S : U \times_M S \to S$.

2. The closed points of M are precisely the $\mathbb{C}$-valued points, as M is assumed to be of finite type. These correspond bijectively to $F(\mathbb{C})$.

3. From (1) and (2), for a family $f : \mathcal{X} \to S$, the map j_f is given on closed points by sending $p \in S(\mathbb{C})$ to the closed point of M corresponding to (the isomorphism class of) the fiber $\mathcal{X}_p$.

Definition 1 *Suppose F is a moduli functor representable by an object M. Then M is called a fine moduli space for F. The family $u : U \to M$ is called a universal family.*

This is too ambitious and it almost never happens when the objects have automorphisms as the functor does not keep track of those at all. For example, $\mathcal{M}_g$ is not representable. There are families with isomorphic fibers but the family itself is not isomorphic to the trivial product family. For if $f : \mathcal{X} \to S$ is such a nontrivial family, all of whose fibers are isomorphic, then point 3 above would imply that the map j_f is a constant map, the same as that for a trivial product family, a contradiction.

We could deal with the coarse moduli space instead, which is a much weaker concept as it does not require a universal family to exist. This is the 'smallest' space that parametrizes the objects of interest.

Definition 2 *For a moduli functor F, M is said to be a coarse moduli space if there is a natural transformation $\xi : F \to h_M$ such that*

(1) For every algebraically closed field Ω, $\xi(\Omega) : F(\Omega) \to h_M(\Omega) = M(\Omega)$ is a bijection.

(2) If $\xi' : F \to h_{M'}$ is another natural transformation for $M' \in \boldsymbol{Sch_f}$, there is a unique map $\psi : M \to M'$ giving a diagram

$$\begin{tikzcd} & h_M \arrow[dd, "\psi"] \\ F \arrow[ur, "\xi"] \arrow[dr, "\xi'"'] & \\ & h_{M'} \end{tikzcd}$$

A coarse moduli space may exist when fine moduli space doesn't. But it is usually harder to work with. For example $\mathcal{M}_g$ has a coarse moduli space M_g, famously constructed by Deligne and Mumford. It may happen that a moduli functor has no coarse moduli space as well.

A less ambitious goal is to study 'infinitesimal' behaviour of the objects of interest and this is the subject of deformation theory.

2 Formal Deformation Theory

We want to study the local behaviour of a moduli functor, even if the moduli functor is not representable. Suppose F is representable by M, an actual geometric object. The local geometry of M at a closed point $m \in M$ (say m corresponds to a geometric object X) is captured by the canonical map $\mathrm{Spec}(\mathcal{O}_{M.m}) \to M$. For example, we can read off the tangent space at m as the morphisms in $M(\mathbb{C}[\epsilon]/\epsilon^2)$ sending the point of $\mathrm{Spec}(\mathbb{C}[\epsilon]/\epsilon^2)$ to m. This is the same as giving a local $\mathbb{C}$-algebra homomorphism $\mathcal{O}_{M.m} \to \mathbb{C}[\epsilon]/\epsilon^2$.

In general, this can be done for any finitely generated Artin local $\mathbb{C}$-algebra A with residue field $\mathbb{C}$, which give the right candidates to analyze local neighbourhoods of $m \in M$. If A is one such, we can consider the morphisms in $M(A)$ sending the unique point of $\mathrm{Spec}A$ to m; these are in bijection with the set $\{\pi : \mathcal{X} \to \mathrm{Spec}A \mid \mathcal{X} \times_A \mathbb{C} \cong X\}$.

Let **Art** be the category of Artin local $\mathbb{C}$-algebras with residue field $\mathbb{C}$, with morphisms as local $\mathbb{C}$-algebra maps.

Definition 3 *A deformation functor is a (covariant) functor Def :* ***Art*** $\to$ ***Set*** *such that Def*$(\mathbb{C}) = \{\bullet\}$ *is a one element set.*

Let $(R, \mathfrak{m}_R), (S, \mathfrak{m}_S)$ be local $\mathbb{C}$-algebras with residue fields $\mathbb{C}$. Recall that the completion of the local ring R at $\mathfrak{m}_R$ is given by the inverse limit

$$\widehat{R} = \varprojlim R/\mathfrak{m}_R^n$$

A local $\mathbb{C}$-algebra morphism $R \to S$ gives a morphism of complete local rings $\widehat{R} \to \widehat{S}$. If S is Artin, the canonical inclusion $S \hookrightarrow \widehat{S}$ becomes an isomorphism (as $\mathfrak{m}_S^n = 0$ for some n). So for $A \in \mathbf{Art}$, the sets $\mathrm{Hom}(R, A)$ and $\mathrm{Hom}(\widehat{R}, A)$ are in bijection.

Example 1 *Let $\mathcal{O}$ be a complete local $\mathbb{C}$-algebra with residue field $\mathbb{C}$. Then $Def_{\mathcal{O}} : \textbf{\textit{Art}} \to \textbf{\textit{Set}}$ defined by $Def_{\mathcal{O}}(A) = Hom(\mathcal{O}, A)$ is a deformation functor.*

Let F be a moduli functor having a fine moduli space M and $m \in M$ a closed point corresponding to an object X. Let $\mathrm{Def}_{F,m}$ be the corresponding deformation functor of X, which sends $A \in \mathbf{Art}$ to the morphisms in $M(A)$ which send the point of $\mathrm{Spec}A$ to m. Then $\mathrm{Def}_{F,m}$ is isomorphic to $\mathrm{Hom}(\widehat{\mathcal{O}_{M,m}}, -)$. The problem is that $\widehat{\mathcal{O}_{M,m}}$ is usually not in $\mathbf{Art}$, so this functor is not quite representable, but close.

Definition 4 *A deformation functor Def is called prorepresentable if there is a complete local $\mathbb{C}$-algebra $\mathcal{O}$ with residue field $\mathbb{C}$ and an isomorphism η of functors $\eta : Hom(\mathcal{O}, \cdot) \to Def(\cdot)$.*

Consider deformations of a nonsingular variety X. The deformation functor $\mathrm{Def}_X : \mathbf{Art} \to \mathbf{Set}$ is given by defining $\mathrm{Def}_X(A)$ to be (the isomorphism classes of) families $\pi : \mathcal{X} \to \mathrm{Spec}A$ where π is flat and a map $\phi : X \to \mathcal{X}$ (necessarily a closed immersion) fitting into a diagram

$$\begin{tikzcd} X \arrow[r, "\phi"] \arrow[d] & \mathcal{X} \arrow[d, "\pi"] \\ \mathrm{Spec}\mathbb{C} \arrow[r, hook] & \mathrm{Spec}A \end{tikzcd}$$

and $X \cong \mathcal{X} \times_A \mathbb{C}$ via the induced fiber product map. A first order deformation of X is an element of $\mathrm{Def}_X(\mathbb{C}[\epsilon]/\epsilon^2)$. The set $\mathrm{Def}_X(\mathbb{C}[\epsilon]/\epsilon^2)$ is called the tangent space of Def_X. Elements of $\mathrm{Def}_X(A)$ are called infinitesimal deformations of X.

Example 2 *Let X be a nonsingular variety. The first order deformations of X are in one-to-one correspondence with the cohomology group $H^1(X, TX)$ where TX is the tangent sheaf. The element $0 \in H^1(X, TX)$ corresponds to the trivial family $X \times_{\mathbb{C}} \mathbb{C}[\epsilon]/\epsilon^2$.*

Example 3 *Let X be a scheme of finite type and $\mathcal{L}$ be a line bundle on X. The first order deformations of $\mathcal{L}$ are in one-to-one correspondence with $H^1(X, \mathcal{O}_X)$. The element $0 \in H^1(X, \mathcal{O}_X)$ corresponds to the trivial extension $p^*\mathcal{L}$, where $p : X \times_{\mathbb{C}} \mathbb{C}[\epsilon]/\epsilon^2 \to X$ is the projection.*

Recall the infinitesimal lifting property: Suppose $(R, \mathfrak{m})$ is a regular local ring and $t : A' \to A$ is a surjection of Artin local $\mathbb{C}$-algebras with $(\ker t)^2 = 0$. Then a map $j : R \to A$ can be lifted to a map $j' : R \to A'$ as in the diagram

$$\begin{tikzcd} & A' \arrow[d, "t"] \\ R \arrow[ur, dashed, "j'"] \arrow[r, "j"'] & A \end{tikzcd}$$

The infinitesimal lifting property has the following implication for smoothness: Let M be a scheme of finite type. Then M is smooth iff the following property holds: For every map $i : T = \mathrm{Spec}A \to T' = \mathrm{Spec}A'$ of affine schemes (where A, A' are not necessarily Artin) with $[\ker(A' \to A)]^2 = 0$ and a map $f : T \to M$, it can be extended to $f' : T' \to M$.

$$\begin{tikzcd} T' \arrow[dr, dashed, "f'"] & \\ T \arrow[u, "i"] \arrow[r, "f"'] & M \end{tikzcd}$$

If M is actually a fine moduli space for a moduli functor F and $m \in M$ a closed point, checking that M is smooth at m is equivalent to checking whether a morphism $f : \mathrm{Spec}A \to M$ for $A \in \mathbf{Art}$ sending the point of $\mathrm{Spec}A$ to m extends to $f' : \mathrm{Spec}A' \to M$, for $A' \in \mathbf{Art}$ a square-zero extension of A. Restated in other words, M is smooth at m iff for every square-zero surjection $A' \to A$ in $\mathbf{Art}$, $\mathrm{Def}_{F,m}(A') \to \mathrm{Def}_{F,m}(A)$ is surjective.

Definition 5 *Let Def be a deformation functor. Def is called smooth if for every square-zero surjection $A' \to A$ in **Art**, Def$(A') \to$ Def(A) is surjective.*

For something weaker, we may ask whether a particular deformation can be extended. This is the notion of (un)obstructedness.

Definition 6 *Let Def be a deformation functor and $\mathcal{X} \in$ Def(A) be an infinitesimal deformation for $A \in$ **Art**. $\mathcal{X}$ is called unobstructed if, for each surjection $A' \to A$ in **Art**, there exists $\mathcal{X}' \in$ Def(A') such that $\mathcal{X}' \mapsto \mathcal{X}$ under Def$(A') \to$ Def(A). If all infinitesimal deformations of $\bullet \in$ Def$(\mathbb{C}) = \{\bullet\}$ are unobstructed, $\bullet$ is called unobstructed, otherwise it is called obstructed.*

$$\begin{tikzcd} \bullet \arrow[r] \arrow[d] & \mathcal{X} \arrow[r] \arrow[d, "\pi"'] & \mathcal{X}' \arrow[d, "\pi'(?)"] \\ \mathrm{Spec}\mathbb{C} \arrow[r, hook] & \mathrm{Spec}A \arrow[r, hook] & \mathrm{Spec}A' \end{tikzcd}$$

Example 4 *For the functor Def$_X$ of a nonsingular variety X, its deformations are unobstructed if the cohomology $H^2(X, TX)$ vanishes. Hence, a nonsingular curve is unobstructed.*

Computing the cohomologies to find the dimension of moduli spaces is not hard: For a curve X of genus g, $h^1(X, TX)$ is 0 for $g = 0$, 1 for $g = 1$ and $3g-3$ for $g \geq 2$.

Warning: If $H^1(X, TX) = 0$, this only means X has no non-trivial first order deformations. This does not say anything about the global deformations. However, if X is nonsingular, it is true that X has no non-trivial first order deformations iff X has no non-trivial infinitesimal deformations.

The category **Art** is closed under fiber products. That is, given maps $A \to B$ and $C \to B$ in **Art**, the fiber product of sets $A \times_B C$ with coordinate wise addition and multiplication is again in **Art**. The (nilpotent) maximal ideal of $A \times_B C$ is $\mathfrak{m}_A \times_{\mathfrak{m}_B} \mathfrak{m}_C$ where $\mathfrak{m}_A, \mathfrak{m}_B, \mathfrak{m}_C$ are the respective maximal ideals.

For a finite dimensional $\mathbb{C}$-vector space V, the object $\mathbb{C}[V] \in \mathbf{Art}$ is given by $\mathbb{C} \oplus V$ as a vector space with multiplication

$$(a, v)(a', v') = (aa', a'v + av')$$

If Def is a deformation functor with the extra condition that the induced fiber product map (We will assume this condition henceforth)

$$\mathrm{Def}(\mathbb{C}[V] \times_{\mathbb{C}} \mathbb{C}[W]) \to \mathrm{Def}(\mathbb{C}[V]) \times_{F(\mathbb{C})=\{\bullet\}} \mathrm{Def}(\mathbb{C}[W])$$

is a bijection for vector spaces V and W, then $\mathrm{Def}(\mathbb{C}[V])$ gets a canonical vector space structure. Considering the addition $+ : \mathbb{C}[V] \times_{\mathbb{C}} \mathbb{C}[V] \to \mathbb{C}[V]$, applying Def and using the bijection gives an addition on $\mathrm{Def}(\mathbb{C}[V])$. The scalar multiplication by $c : \mathbb{C}[V] \to \mathbb{C}[V]$ gives the scalar multiplication on $\mathrm{Def}(\mathbb{C}[V])$. In particular, the tangent space of a deformation functor is a vector space.

3 Formal Families

For a moduli functor, the best possible case is representability. For a deformation functor Def, the best possible case is prorepresentability. Since this is also not possible every time, we need weaker notions.

Let $\widehat{\mathbf{Art}}$ be the category of complete local $\mathbb{C}$-algebras with residue field $\mathbb{C}$. Instead of isomorphisms, consider only natural transformations

$$\eta : \mathrm{Hom}(\mathcal{O}, -) \to \mathrm{Def}$$

for $\mathcal{O}$ in $\widehat{\mathbf{Art}}$. An easy observation is that Def (any functor really) can be extended to $\widehat{\mathrm{Def}} : \widehat{\mathbf{Art}} \to \mathbf{Set}$ by defining

$$\widehat{\mathrm{Def}}(\mathcal{O}) := \varprojlim \mathrm{Def}(\mathcal{O}/\mathfrak{m}^n)$$

where $\mathfrak{m}$ is the maximal ideal in $\mathcal{O}$.

An element $\hat{u}$ of $\widehat{\mathrm{Def}}(\mathcal{O})$ is called a formal element of Def. This is a sequence of compatible elements $u_n \in \mathrm{Def}(\mathcal{O}/\mathfrak{m}^n)$, so should be thought of as a sequence of infinitesimal deformations, one for each Artin ring $\mathcal{O}/\mathfrak{m}^n$, where the compatibility is given by pullbacks.

Theorem 1 *(Formal Yoneda) There is a natural one-to-one correspondence between $\widehat{Def}(\mathcal{O})$ and morphisms $\eta : Hom(\mathcal{O}, -) \to Def$.*

A quick sketch of the correspondence: A morphism η defines a formal element $\hat{u} \in \widehat{\mathrm{Def}}(\mathcal{O})$ by setting u_n to be the image of $\pi_n \in \mathrm{Hom}(\mathcal{O}, \mathcal{O}/\mathfrak{m}^n)$ in $\mathrm{Def}(\mathcal{O}/\mathfrak{m}^n)$, where π_n is the canonical projection. In the other direction, a formal element $\hat{u}$ given by a sequence $u_n \in \mathrm{Def}(\mathcal{O}/\mathfrak{m}^n)$ defines a morphism η as follows: By the ordinary Yoneda lemma, each u_n defines a morphism $\eta_n : \mathrm{Hom}(\mathcal{O}/\mathfrak{m}^n, -) \to \mathrm{Def}$. Let $f \in \mathrm{Hom}(\mathcal{O}, A)$. Since A is Artin, f factors through $\mathcal{O}/\mathfrak{m}^n$ for large enough n, say $f = f_n \circ \pi_n$, $f_n : \mathcal{O}/\mathfrak{m}^n \to A$. Define $\eta(A)(f) := \eta_n(A)(f_n)$. More concretely, $\eta_{n+1} \circ \pi_{n+1,n} = \eta_n$ so η is the limit $\varinjlim \eta_n$.

$$\begin{tikzcd}
\mathrm{Hom}(\mathcal{O}/\mathfrak{m}^n, -) \arrow[rr, "\pi_{n+1,n}"] \arrow[dr, "\eta_n"'] & & \mathrm{Hom}(\mathcal{O}/\mathfrak{m}^{n+1}, -) \arrow[dl, "\eta_{n+1}"] \\
& \mathrm{Def} &
\end{tikzcd}$$

If Def is prorepresentable with $\eta : \mathrm{Hom}(\mathcal{O}, -) \to \mathrm{Def}$ an isomorphism for $\mathcal{O} \in \widehat{\mathbf{Art}}$, by formal Yoneda there is a ‘formal family’ $\hat{u} \in \widehat{\mathrm{Def}}(\mathcal{O})$ inducing the isomorphism η. Such a family $\hat{u}$ is called a universal formal family for Def.

If F is a moduli functor with a fine moduli space M and universal family $u : \mathcal{X} \to M$ and for a point $m \in M$, if $\mathrm{Def}_{F,m}$ is the corresponding deformation functor, it is prorepresentable by $\mathcal{O} = \widehat{\mathcal{O}_{M,m}}$. The universal formal family $\hat{u} = \{u_n\}_{n \geq 1}$ is given by pulling back u.

$$\begin{tikzcd}
\mathcal{X}_n \arrow[r] \arrow[d, "u_n"'] & \mathcal{X} \arrow[d, "u"] \\
\mathrm{Spec}(\mathcal{O}/\mathfrak{m}_m^n) \arrow[r] & M
\end{tikzcd}$$

The following criterion for prorepresentability is due to Schlessinger.

Theorem 2 *Let Def be a deformation functor. For $A \to B$, $C \to B$ in **Art**, consider the natural map*

$$\mathit{Def}(A \times_B C) \to \mathit{Def}(A) \times_{\mathit{Def}(B)} \mathit{Def}(C) \qquad (\star)$$

Def is prorepresentable if the following properties hold:
(1) $(\star)$ is surjective whenever $C \to B$ is small.
(2) $(\star)$ is bijective when $B = \mathbb{C}$ and $A = C = \mathbb{C}[\epsilon]/\epsilon^2$.
(3) The tangent space $\mathit{Def}(\mathbb{C}[\epsilon]/\epsilon^2)$ is a finite dimensional vector space.
(4) $(\star)$ is bijective if $A = C$, the maps $A \to B$, $C \to B$ are equal & small.

In general, the deformation functor Def_X of a projective nonsingular variety X is not prorepresentable, but it satisfies the first three conditions of Schlessinger's theorem.

There is a notion of smooth morphisms of functors.

Definition 7 *Let $\xi : \mathit{Def}_1 \to \mathit{Def}_2$ be a morphism of deformation functors. ξ is called smooth if for every surjection $A \to B$ in **Art**, the map induced by the fiber product $\mathit{Def}_1(A) \to \mathit{Def}_1(B) \times_{\mathit{Def}_2(B)} \mathit{Def}_2(A)$ is surjective.*

The motivation for this definition is the fact: If $\mathcal{O}, \mathcal{O}' \in \widehat{\mathbf{Art}}$ and $\phi : \mathcal{O} \to \mathcal{O}'$ is a morphism, the induced morphism of deformation functors $\mathrm{Hom}(\mathcal{O}', -) \to \mathrm{Hom}(\mathcal{O}, -)$ is smooth iff $\mathcal{O}' = \mathcal{O}[[z_1, z_2, \cdots, z_n]]$, that is $\mathcal{O}'$ is a formal power series ring over $\mathcal{O}$.

Apply the definition of smoothness to $A = C = \mathbb{C}[\epsilon]/\epsilon^2$ and $B = \mathbb{C}$ to get that $\xi(\mathbb{C}[\epsilon]/\epsilon^2) : \mathrm{Def}_1(\mathbb{C}[\epsilon]/\epsilon^2) \to \mathrm{Def}_2(\mathbb{C}[\epsilon]/\epsilon^2)$ is surjective. This map is called the differential and is a linear map of vector spaces.

Definition 8 *Let Def be a deformation functor. If there exists an object $\mathcal{O} \in \widehat{\boldsymbol{Art}}$ and a formal family $\hat{u} \in \widehat{Def}(\mathcal{O})$ with the property that the map defined by $\hat{u}$ by the formal Yoneda lemma $Hom(\mathcal{O}, -) \to Def$ is smooth, Def is called versal. The pair $(\mathcal{O}, \hat{u})$ is called a versal pair.*

If $(\mathcal{O}, \hat{u})$ is a versal pair, taking the surjection $A \to \mathbb{C}$ in **Art** shows that $\mathrm{Hom}(\mathcal{O}, A) \to \mathrm{Def}(A)$ is surjective for all A and from this it follows that $\mathrm{Hom}(\mathcal{O}, R) \to \widehat{\mathrm{Def}}(R)$ is surjective for all $R \in \widehat{\mathbf{Art}}$. So versality means that every formal element $\hat{v} \in \widehat{\mathrm{Def}}(R)$ is induced by pulling back $\hat{u}$ on $\mathrm{Spec}\mathcal{O}$ by a (not necessarily unique) morphism $\mathcal{O} \to R$.

Definition 9 *A versal pair $(\mathcal{O}, \hat{u})$ for Def is called semiuniversal if in addition, the differential $Hom(\mathcal{O}, \mathbb{C}[\epsilon]/\epsilon^2) \to Def(\mathbb{C}[\epsilon]/\epsilon^2)$ is an isomorphism.*

Examples of deformation functors admitting a semiuniversal pair include the Picard functor of deformations of a line bundle on a fixed scheme X and the deformation functor of a projective variety X.

4 A Modern Viewpoint: DGLAs

The modern viewpoint to deformation theory by the giants in mathematics (Deligne, Drinfeld, Quillen and more) is that every deformation problem over a field of characteristic zero is governed by a differential graded Lie algebra (or DGLA for short). The Maurer-Cartan locus gives the infinitesimal deformations of an algebro-geometric object, as will be explained shortly. The prototypical example of this phenomenon is the deformation theory of a compact complex manifold, which arises from the Kodaira-Spencer DGLA.

Definition 10 *A DGLA over a field k consists of*
(1) A graded k-vector space $L = \bigoplus_{i \geq 0} L^i$
(2) For each $i \geq 0$, a differential $d : L^i \to L^{i+1}$, such that $d^2 = 0$
(3) A bilinear Lie bracket $[\cdot, \cdot] : L \times L \to L$ with $[L^i, L^j] \subset L^{i+j}$
satisfying the following properties for homogeneous elements a, b, c of degrees $|a|, |b|, |c|$ respectively:

- *Graded antisymmetry: $[a, b] = -(-1)^{|a||b|}[b, a]$*
- *Graded Leibniz rule: $d[a, b] = [da, b] + (-1)^{|a|}[a, db]$*

- *Graded Jacobi identity:*

$$(-1)^{|a||c|}[a,[b,c]] + (-1)^{|b||a|}[b,[c,a]] + (-1)^{|c||b|}[c,[a,b]] = 0$$

Definition 11 *Let L be a DGLA. The Maurer-Cartan locus of L is the set*

$$MC(L) := \left\{ x \in L^1 \mid dx + \frac{1}{2}[x,x] = 0 \right\}$$

We take most of what follows without all the details.

Example 5 *Let L be a DGLA and suppose $\mathfrak{m}$ is the maximal ideal of $A \in \boldsymbol{Art}$. The tensor product $L \otimes \mathfrak{m} = \bigoplus_{i \geq 0} L^i \otimes \mathfrak{m}$ gets a structure of a DGLA with differential $d(x \otimes r) = dx \otimes r$ and bracket $[x \otimes r, y \otimes s] = [x,y] \otimes rs$, both extended linearly.*

Recall that $\mathfrak{m}^n = 0$ for some n. If $a \in L^0 \otimes \mathfrak{m}$, then $\mathrm{ad}_a = [a, -]$ is a derivation as is easy to verify and is nilpotent as $\mathfrak{m}$ is. Hence $\exp(\mathrm{ad}_a)$ is an automorphism of $L \otimes \mathfrak{m}$.

There is a 'gauge action' of $L^0 \otimes \mathfrak{m}$ on $L \otimes \mathfrak{m}$ given by: if $a \in L^0 \otimes \mathfrak{m}$, define

$$e^a \cdot x := x + \sum_{n=0}^{\infty} \frac{\mathrm{ad}_a^n}{(n+1)!} (\mathrm{ad}_a(x) - da)$$

The relation $e^b \cdot (e^a \cdot x) = e^{b \bullet a} \cdot x$ holds, where $b \bullet a$ is the Baker-Campbell-Hausdorff product in $L^0 \otimes \mathfrak{m}$. Hence the gauge action is really an action. The Maurer-Cartan Locus of $L \otimes \mathfrak{m}$ is invariant under this action, therefore we can consider the set $\frac{MC(L \otimes \mathfrak{m})}{\text{gauge equivalence}}$.

Definition 12 *Let L be a DGLA. The deformation functor associated to L, $Def_L : \boldsymbol{Art} \to \boldsymbol{Set}$ is*

$$Def(A) = \frac{MC(L \otimes \mathfrak{m}_A)}{\textit{gauge equivalence}}$$

where $\mathfrak{m}_A$ is the maximal ideal of A.

Principle 1 *Let $Def \colon \boldsymbol{Art} \to \boldsymbol{Set}$ be a deformation functor. Then there exists a DGLA L such that $Def \cong Def_L$.*

The DGLA L above will not be unique, but it will be unique upto quasi-isomorphism. Two DGLAs $(L, d_L, [\cdot,\cdot]_L)$ and $(M, d_M, [\cdot,\cdot]_M)$ are called quasi-isomorphic if there is a Lie algebra homomorphism $\phi = \{\phi_i\}_{i \geq 0}$ of complexes such that ϕ induces an isomorphism the cohomology spaces $H^i(L, d_L) \cong H^i(M, d_M)$ for all $i \geq 0$.

Let X be a compact complex manifold of dimension n. There is the Kodaira-Spencer DGLA associated to X,

$$KS(X) := A^{0,*}(TX^{1,0}) = \bigoplus_{q=0}^{n} A^{0,q}(TX^{1,0})$$

where $TX^{1,0}$ is the holomorphic tangent bundle of X. The differential of $KS(X)$ is the Dolbeault differential of $TX^{1,0}$ and the Lie bracket is given by the wedge product on the forms and the usual Lie bracket on the tangent vector fields.

For a variety X, an infinitesimal deformation of X over an $A \in \mathbf{Art}$ is equivalent from the definition, to giving a sheaf $\mathcal{O}_{\mathcal{X}}$ of A-algebras and the data of a morphism of sheaves $\mathcal{O}_{\mathcal{X}} \to \mathcal{O}_X$ with $\mathcal{O}_{\mathcal{X}}$ flat over A and the induced map $\mathcal{O}_{\mathcal{X}} \otimes_A \mathbb{C} \to \mathcal{O}_X$ is an isomorphism. We use this equivalent definition for a complex manifold, with $\mathcal{O}_X$ now being the sheaf of holomorphic functions. Let Def_X be the deformation functor of X.

Theorem 3 *For a compact complex manifold X, there is an isomorphism*

$$Def_{KS(X)} \to Def(X)$$

The Maurer-Cartan condition really comes up here as the integrability condition of the Newlander-Nirenberg theorem.

References

[1] Robin Hartshorne. *Algebraic Geometry*. Springer, 1997.

[2] Robin Hartshorne. *Deformation Theory*. Springer, 2009.

[3] Marco Manetti. Differential graded lie algebras and formal deformation theory.

[4] Marco Manetti. Lectures on deformations of complex manifolds.

[5] M. Schlessinger. Functors of artin rings. *Transactions of the American Mathematical Society*, 130:208–222, 1968.

[6] Edoardo Sernesi. *Deformations of Algebraic Schemes*. Springer, 2006.