

Non-Simple Forbidden Configurations and Design Theory

Richard Anstee
Farzin Barekat
UBC, Vancouver

University of Vermont, June 2008

A survey of results about forbidden configurations is available at my web page and I would welcome comments. The results in this paper are examples of some of the exact bounds that have been obtained. A listing of many exact bounds, asymptotic bounds and open problems can be found in:

Survey at www.math.ubc.ca/~ansteer

A matrix is **simple** if it is a $(0,1)$ -matrix with no repeated columns.

A matrix is **simple** if it is a $(0,1)$ -matrix with no repeated columns.

e.g. K_m^d is the $m \times \binom{m}{d}$ simple matrix
of all columns of column sum d .

Design Theory

Definition A **triple system** $S_\lambda(2, 3, m)$ is a collection of triples, subsets of $\{1, 2, \dots, m\}$, such that for each pair $i, j \in \{1, 2, \dots, m\}$, there are exactly λ triples containing i, j .

Design Theory

Definition A **triple system** $S_\lambda(2, 3, m)$ is a collection of triples, subsets of $\{1, 2, \dots, m\}$, such that for each pair $i, j \in \{1, 2, \dots, m\}$, there are exactly λ triples containing i, j .

A triple system is **simple** if there are no repeated triples

Theorem (Dehon, 1983) Let m, λ be given. Assume $m \geq \lambda + 2$ and $m \equiv 1, 3 \pmod{6}$. Then there exists a simple triple system $S_\lambda(2, 3, m)$.

Theorem (Dehon, 1983) Let m, λ be given. Assume $m \geq \lambda + 2$ and $m \equiv 1, 3 \pmod{6}$. Then there exists a simple triple system $S_\lambda(2, 3, m)$.

Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple triple system $S_\lambda(2, 3, m)$.

Theorem (Dehon, 1983) Let m, λ be given. Assume $m \geq \lambda + 2$ and $m \equiv 1, 3 \pmod{6}$. Then there exists a simple triple system $S_\lambda(2, 3, m)$.

Let $T_{m,\lambda}$ denote the element-triple incidence matrix of a simple triple system $S_\lambda(2, 3, m)$.

Thus $T_{m,\lambda}$ is an $m \times \frac{\lambda}{3} \binom{m}{2}$ simple matrix with all columns of column sum 3 and having no submatrix

$$J_{2,\lambda+1} = \begin{bmatrix} 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

$\lambda+1$

Theorem Let A be an $m \times n$ simple matrix with no submatrix

$$J_{2,q} = \begin{bmatrix} \overbrace{1 & 1 & \cdots & 1}^q \\ 1 & 1 & \cdots & 1 \end{bmatrix}$$

Then

$$n \leq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-2}{3} \binom{m}{2}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,q-2}]$$

if $m \geq q$ and $m \equiv 1, 3 \pmod{6}$.

Theorem Let q be given. Then there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{1,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 & 1 & \cdots & 1}^q \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Then

$$n \leq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-2}{3} \binom{m}{2} + \binom{m}{m}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,q-2} K_m^m]$$

if $m \geq q$ and $m \equiv 1, 3 \pmod{6}$.

Why do we need $m > M$?

If $m = q + 1$, we may take

$$A = [K_m^0 K_m^1 K_m^2 K_m^3 \dots K_m^m]$$

which has

$$\binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \dots + \binom{m}{m}$$

columns and which has no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{1,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 & 1 & \dots & 1}^q \\ 1 & 1 & \dots & 1 \\ 0 & 0 & \dots & 0 \end{bmatrix}$$

but has

$$\frac{(q+1)q}{6}$$

more columns than the previous bound.

Theorem Let q be given. Then there exists an M so that for $m > M$, if A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{2,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 & 1 & \cdots & 1}^q \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Then

$$n \leq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-3}{3} \binom{m}{2} + \binom{m}{m-2} + \binom{m}{m-1} + \binom{m}{m}$$

with equality only for

$$A = [K_m^0 K_m^1 K_m^2 T_{m,a} T_{m,b}^c K_m^{m-2} K_m^{m-1} K_m^m]$$

(for some choice a, b with $a + b = q - 3$)

if $m \geq q$ and $m \equiv 1, 3 \pmod{6}$.

Proof Ideas

Assume that A is an $m \times n$ simple matrix with no submatrix which is a row permutation of

$$\begin{bmatrix} J_{2,q} \\ 0_{2,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 & 1 & \cdots & 1}^q \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Assume n is larger than the given bound:

$$n > \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \frac{q-3}{3} \binom{m}{2} + \binom{m}{m-2} + \binom{m}{m-1} + \binom{m}{m}$$

Pigeonhole Argument

Idea Total number of $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$'s $\leq 6(q-1)\binom{m}{4}$.

A column of sum k yields $\binom{k}{2}\binom{m-k}{2}$ $\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$'s.

Let a_i = number of columns of column sum i

Thus

$$\binom{m-2}{2}a_2 + \binom{3}{2}\binom{m-3}{2}a_3 + \cdots + \binom{m-2}{2}a_{m-2} \leq 6(q-1)\binom{m}{4}$$

(a_i = number of columns of column sum i)

If we assume $a_2 + a_3 + \cdots + a_{m-2} \geq \frac{q+3}{3} \binom{m}{2}$, then we deduce

$$2 \binom{m}{2} - c_1 m \leq a_2 + a_{m-2} \leq 2 \binom{m}{2}$$

$$a_4 + a_5 + \cdots + a_{m-4} \leq c_2 m$$

$$a_3 + a_{m-3} \geq \frac{q-3}{3} \binom{m}{2} - c_3 m$$

These bounds already bring us to within some constant times m of our desired bound.

Degrees

Consider the columns of column sum 3.

Let $d_1(ij) = \text{degree of } \begin{matrix} i \\ j \end{matrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{array}{cccc}
 & & \overbrace{\hspace{1.5cm}}^{d_1(ij)} & \\
 i & 1 & 1 & \cdots & 1 \\
 j & 1 & 1 & \cdots & 1 \\
 & 1 & & & \\
 & & 1 & & \\
 & & & \ddots & \\
 & & & & 1
 \end{array}$$

Thus $d_1(ij) \leq q - 1$.

Consider the columns of column sum $m - 3$.

Let $d_0(ij) = \text{degree of } \begin{matrix} i \\ j \\ 0 \end{matrix} \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

$$\begin{array}{cccc}
 & \overbrace{\hspace{10em}}^{d_0(ij)} & & \\
 i & 0 & 0 & \cdots & 0 \\
 j & 0 & 0 & \cdots & 0 \\
 & 0 & & & \\
 & & 0 & & \\
 & & & \ddots & \\
 & & & & 0
 \end{array}$$

Thus $d_0(ij) \leq q - 1$.

One can show much more:

$$d_0(ij) + d_1(ij) \leq q - 1$$

$$\begin{array}{cccccccc}
 & & & \overbrace{\hspace{4em}}^{d_1(ij)} & & & & \\
 i & 1 & 1 & \dots & 1 & ? & ? & \dots & ? \\
 j & 1 & 1 & \dots & 1 & ? & ? & \dots & ? \\
 & 1 & & & & & & & \\
 & & 1 & & & & & & \\
 & & & \ddots & & 0 & & & \\
 & & & & 1 & & 0 & & \\
 & & & & & & & \ddots & \\
 & & & & & & & & 0 \\
 k & * & * & \dots & * & 0 & 0 & \dots & 0 \\
 l & * & * & \dots & * & 0 & 0 & \dots & 0 \\
 & & & & & \underbrace{\hspace{4em}}_{d_0(kl)} & & &
 \end{array}$$

Given i, j , for most choices k, l , we have that the entries '*' are 0's and the entries '?' are 1's and so, assuming $a_2 = a_{m-2} = \binom{m}{2}$

$$d_1(ij) + d_0(kl) \leq q - 3$$

Fact: $\sum_{ij} d_0(ij) + d_1(ij) = 3a_3 + 3a_{m-3} \approx (q-3) \binom{m}{2}$

We have

$$d_1(ij) + d_0(kl) \leq q - 3$$

$$d_0(ij) + d_1(kl) \leq q - 3$$

Idea: if for any choice i, j we have

$$d_0(ij) + d_1(ij) \geq q - 2$$

then for 'most' pairs k, l we have

$$d_0(kl) + d_1(kl) \leq q - 4$$

which then violates our estimate

$$\sum_{ij} d_0(ij) + d_1(ij) \approx (q-3) \binom{m}{2}$$

Turán's Bound

We can show that the number of pairs i, j with $d_0(ij) + d_1(ij) = q - 3$ is greater than $\binom{m}{2} - c_4 m$.

Turán's Bound

We can show that the number of pairs i, j with $d_0(ij) + d_1(ij) = q - 3$ is greater than $\binom{m}{2} - c_4 m$.

Idea: By Turán's argument, there is a large 'clique' of rows B so that given any c_5 , if m is large enough

$$i, j \in B \Rightarrow d_0(ij) + d_1(ij) = q - 3$$

$$|B| \geq c_5 \sqrt{m}$$

We can deduce that in the columns of sum $4, 5, \dots, m-4$, the rows indexed by B avoid certain structures

$$A_{4,5,\dots,m-4} = \left. \begin{array}{l} \text{rows } B \end{array} \right\} \left[\begin{array}{c} \text{no configuration} \\ \left[\begin{array}{c} 1 \\ 1 \\ 0 \\ 0 \end{array} \right] \end{array} \right]$$

The size of B allows us to select from up to $\binom{c_5 \sqrt{m}}{2} \approx \frac{c_5^2}{2} m$ pairs i, j with $d_0(ij) + d_1(ij) = q - 3$.

We finally argue that $a_2 = a_{m-2} = \binom{m}{2}$, $a_4 = a_5 = \dots = a_{m-4} = 0$ and for **all** pairs $i, j \in \{1, 2, \dots, m\}$

$$d_0(ij) + d_1(ij) = q - 3$$

We then can also argue that there exists positive integers a, b with $a + b = q - 3$ so that for **all** pairs i, j

$$d_1(ij) = a, \quad d_0(ij) = b$$

We finally argue that $a_2 = a_{m-2} = \binom{m}{2}$, $a_4 = a_5 = \dots = a_{m-4} = 0$ and for **all** pairs $i, j \in \{1, 2, \dots, m\}$

$$d_0(ij) + d_1(ij) = q - 3$$

We then can also argue that there exists positive integers a, b with $a + b = q - 3$ so that for **all** pairs i, j

$$d_1(ij) = a, \quad d_0(ij) = b$$

Columns of column sum 3 yield simple triple system with $\lambda = a$

We finally argue that $a_2 = a_{m-2} = \binom{m}{2}$, $a_4 = a_5 = \dots = a_{m-4} = 0$ and for **all** pairs $i, j \in \{1, 2, \dots, m\}$

$$d_0(ij) + d_1(ij) = q - 3$$

We then can also argue that there exists positive integers a, b with $a + b = q - 3$ so that for **all** pairs i, j

$$d_1(ij) = a, \quad d_0(ij) = b$$

Columns of column sum 3 yield simple triple system with $\lambda = a$

Columns of column sum $m - 3$ yield complement of simple triple system with $\lambda = b$.

Problem Can we reduce M to something close to $q + 2$?

Problem Let q be given. Does there exist an M so that for $m > M$, if A is an $m \times n$ simple matrix with no $4 \times q$ submatrix which is a row permutation of

$$\begin{bmatrix} J_{3,q} \\ 0_{1,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 & 1 & \cdots & 1}^q \\ 1 & 1 & \cdots & 1 \\ 1 & 1 & \cdots & 1 \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

Then

$$n \leq \binom{m}{0} + \binom{m}{1} + \binom{m}{2} + \binom{m}{3} + \frac{q-3}{4} \binom{m}{3} + \binom{m}{m}$$

with equality only if there exists a simple 3-design $S_\lambda(3, 4, m)$ with $\lambda = q - 2$?

Problem Let q be given. Let A is an $m \times n$ simple matrix with no $4 \times 2q$ submatrix which is a row and column permutation of

$$\begin{bmatrix} J_{2,q} & 0_{2,q} \\ 0_{2,q} & J_{2,q} \end{bmatrix} = \begin{bmatrix} \overbrace{1 \ \cdots \ 1}^q & \overbrace{0 \ \cdots \ 0}^q \\ 1 \ \cdots \ 1 & 0 \ \cdots \ 0 \\ 0 \ \cdots \ 0 & 1 \ \cdots \ 1 \\ 0 \ \cdots \ 0 & 1 \ \cdots \ 1 \end{bmatrix}$$

Can we show that n is $O(m^2)$?

Thanks to SIAM and the University of Vermont for organizing the conference in beautiful Vermont!