Latin Squares and Projective Planes

A *latin square* of order $n$ is an $n \times n$ array $L$ with entries chosen from $\{1, 2, \ldots, n\}$ with the property, for each $i \in \{1, 2, \ldots, n\}$, that each row contains symbol $i$ and each column contains symbol $i$. Alternatively symbol $i$ appears in a set of positions in the array $L$ which form a permutation.

Note that a solution to a sudoku puzzle is a $9 \times 9$ latin square.

A pair of latin squares $L_1 = (a_{ij})$ and $L_2 = (b_{ij})$ of order $n$ are said to be *orthogonal* if for each possible pair $x, y$ for $i, j \in \{1, 2, \ldots, n\}$, there is some choice $i, j$ such that $a_{ij} = x$ and $b_{ij} = y$.

Consider two orthogonal latin squares

\[
L_1 = \begin{bmatrix}
1 & 2 & 3 \\
3 & 1 & 2 \\
2 & 3 & 1 \\
\end{bmatrix},
L_2 = \begin{bmatrix}
1 & 2 & 3 \\
2 & 3 & 1 \\
3 & 1 & 2 \\
\end{bmatrix}
\]

from this we can form an $9 \times 4$ array as follows

\[
T = \begin{bmatrix}
i & j & L_1 & L_2 \\
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 3 & 3 & 3 \\
2 & 1 & 3 & 2 \\
2 & 2 & 1 & 3 \\
2 & 3 & 2 & 1 \\
3 & 1 & 2 & 3 \\
3 & 2 & 3 & 1 \\
3 & 3 & 1 & 2 \\
\end{bmatrix}
\]

This is called an *orthogonal design*. It satisfies what is sometimes called the two finger rule. If you take two fingers and run down two columns of $T$ you encounter all 9 possible pairs of integer $i, j$ where $i, j \in \{1, 2, 3\}$. This being true for columns 1,2 is given. This being true for columns 1,3 (respectively 1,4) follows from the fact that $L_1$ (resp. $L_2$) has each symbol appearing in a column and this being true for columns 2,3 (respectively 2,4) follows from the fact that $L_1$ (resp. $L_2$) has each symbol appearing in a row. This being true for columns 3,4 corresponds to the fact that $L_1, L_2$ are orthogonal.

This procedure and argument works more generally so that if we have $t$ mutually orthogonal latin squares (MOLS) of order $n$, say $L_1, L_2, \ldots L_t$ then we can form an orthogonal design $n^2 \times (t+2)$ from the latin squares as we did above, with the first two columns containing all the $n^2$ pairs $i, j$ is the order as given while the $j + 2$nd column arises from the latin square $L_j$.

When $t = n - 1$ (i.e. we have $n - 1$ MOLS of order $n$), we are able to construct a projective plane of order $n$. Recall that a projective plane of order $n$ has $n^2 + n + 1$ lines each with $n + 1$ points and $n^2 + n + 1$ points each on $n + 1$ lines. Each pair of lines intersects at a unique point and every pair of points determines a unique line. Let me do the construction for the case of the two $3 \times 3$ orthogonal latin squares above.
We construct a $13 \times 13$ $(0,1)$-matrix where the rows are indexed by points and the columns are indexed by lines and we place a 1 in row $i$ and column $j$ if line $j$ has point $i$. The first 4 columns and first four rows may be assumed to have a very standard form as shown. The blue notations are just entries from the orthogonal array and are not part of the $13 \times 13$ $(0,1)$-matrix.

| 1 1 1 1 | 0 0 0 | 0 0 0 | 0 0 0 | 0 0 0 |
| 1 0 0 0 | 1 1 1 | 0 0 0 | 0 0 0 |
| 1 0 0 0 | 0 0 0 | 1 1 1 | 0 0 0 |
| 1 0 0 0 | 0 0 0 | 0 0 0 | 1 1 1 |
| 0 1 0 0 | 1 1 0 0 | 1 1 0 0 | 1 1 0 0 |
| 0 1 0 0 | 2 0 1 0 | 2 0 1 0 | 2 0 1 0 |
| 0 1 0 0 | 3 0 0 1 | 3 0 0 1 | 3 0 0 1 |
| 0 0 1 0 | 1 1 0 0 | 3 0 0 1 | 2 0 1 0 |
| 0 0 1 0 | 2 0 1 0 | 1 1 0 0 | 3 0 0 1 |
| 0 0 1 0 | 3 0 0 1 | 2 0 1 0 | 1 1 0 0 |
| 0 0 0 1 | 1 1 0 0 | 2 0 1 0 | 3 0 0 1 |
| 0 0 0 1 | 2 0 1 0 | 3 0 0 1 | 1 1 0 0 |
| 0 0 0 1 | 3 0 0 1 | 1 1 0 0 | 2 0 1 0 |

Assume that we violate the axioms of a projective plane namely two lines (columns) which intersect at two points forming a $2 \times 2$ submatrix $[11]$ as shown below.

We can assume the lines come from different sets of columns corresponding to two different columns of the orthogonal design (but not its first column) which we have identified as $r, s$. Thus column $i$ is one of the (three) columns arising from column $r$ of the orthogonal design. Then the 1’s in column $i$ of the (three) rows identified as coming from column $r$ of the orthogonal array, correspond to symbol $i$ (it depends on the column, the first column corresponds to 1, the second column corresponds to 2 etc. ). Similarly the 1’s in column $j$ of the (three) rows identified as coming from column $s$ of the orthogonal array correspond to symbol $j$. Thus positions $(c, r)$ and $(d, s)$ in $T$ have symbol $i$ and positions $(d, r)$ and $d, s$ in $T$ have symbol $j$. This violates the orthogonality of columns $r$ and $s$ (two finger rule) in $T$.

This easily extends to larger orthogonal latin squares of order $n$ (replace three by $n$) and their associated orthogonal array.

For our particular example, we can read of the 13 lines from our $13 \times 13$ matrix as

$\{1, 2, 3, 4\}$, $\{1, 5, 6, 7\}$, $\{1, 8, 9, 10\}$, $\{1, 11, 12, 13\}$

$\{2, 5, 8, 11\}$, $\{2, 6, 9, 12\}$, $\{2, 7, 10, 13\}$
\{3, 5, 9, 13\}, \{3, 6, 10, 11\}, \{3, 7, 8, 12\}
\{4, 5, 10, 12\}, \{4, 6, 8, 13\}, \{4, 7, 9, 11\}