

Special DeBruijn Sequence that yields a Card Trick

This example was shown to me by Ron Graham. It is a special deBruijn sequence of 32 bits (to be thought of as a cyclic sequence) that can be generated in an easy way.

00001010111011000111110011010010

The sequence has the property that each 5 bit sequence appears exactly once as a sequence of 5 consecutive bits in the sequence. The recurrence that generates the sequence is relatively easy

$$x_k = \begin{cases} 0 & \text{if } x_k = 1 \text{ and } x_{k+1} = x_{k+2} = x_{k+3} = x_{k+4} = 0 \\ 1 & \text{if } x_k = x_{k+1} = x_{k+2} = x_{k+3} = x_{k+4} = 0 \\ x_k + x_{k+3} \pmod{2} & \end{cases}$$

The idea behind having the recurrence is that given 5 consecutive bits in the sequence then in ‘relatively little work’ you can compute the next 4 bits.

Now we create a deck of 32 cards consisting of A,2,3,4,5,6,7 in each of the 5 suits and put them in a special order corresponding to the sequence. We encode the suits as follows

00	♣
01	♠
10	◇
11	♡

We encode the ranks simply enough by subtracting 1:

000	A	100	5
001	2	101	6
010	3	110	7
011	4	111	8

Now we look at the first 5 bits of the sequence 00001 and think of it as 00 plus 001 and then it becomes ♣2. Then the next 5 bit sequence becomes 00010 is 00 plus 010 and so ♣3. Then 00101 is 00 plus 101 and so is ♣6. Then 01010 is 01 plus 010 and so is ♠3. And then 10101 is 10 plus 101 and so is ◇6. Etc.

The full (cyclic sequence) is

♣2, ♣3, ♣6, ♠3, ◇6, ♠4, ◇8, ♠7♡6, ♡4, ◇7, ♠5, ♡A, ◇2, ♣4, ♣8,
 ♠8, ♡8, ♡7, ♡5, ♡2, ◇4, ♣7, ♠6, ♡3, ◇5, ♠2, ◇3, ♣5, ♠A, ◇A, ♣A

Now to make a card trick we show the deck to the audience and they see that it is in a ‘random’ order. We allow the deck to be cut a number of times (with supervision!) which preserves the cyclic order. Now we deal out 5 cards in order. An audience member looks at the cards and tells us which cards are red say the first, third and fourth. Thus the five cards come from the 5 positions in the cyclic sequence 10110 since red cards have a 1 initially and black cards have a 0. We compute quietly and hopefully accurately the next 4 bits using the recurrence as 0011. Be prepared to be fooled by the two special cases of the recurrence. Thus the five cards are encoded as 10110, 01100, 11000, 10001, and 00011 and hence are the cards in order are ◇7 (always remember to add one to get the rank!), ♠5, ♡A, ◇2, ♣4. Your audience will be suitably amazed.

The take home message is that 5 bits of information (in this case the positions of the red cards if any) potentially enables you to distinguish $2^5 = 32$ possibilities which for us is the position in the sequence of the top card after cutting.