MATH 443 Various Notations

In this course, a graph \( G \) consists of a finite set \( V(G) = V \) of vertices and a finite set \( E(G) = E \) of edges such that each edge \( e = uv \) has associated with it two endpoints \( u, v \in V \) which need not be distinct. A loop is an edge with both endpoints the same e.g. \( e = vv \).

A simple graph \( G \) (or just a graph \( G \)) has no multiple edges or loops. We refer to a multigraph if we allow multiple edges and a general graph is we also allow loops.

A walk of length \( k \) is a sequence \( v_0e_1v_1e_2v_2\cdots e_kv_k \) such that \( v_i \in V(G) \), \( e_i \in E(G) \) and the endpoints of \( e_i \) are \( v_{i-1}, v_i \) or in other words \( e_i = v_{i-1}v_i \). We say that \( W \) is a \( v_0-v_k \)-walk.

A trail is a walk with no repeated edges.

A path is a walk with no repeated vertices.

A \( u-v \)-walk (respectively trail or path) is a walk (respectively trail or path) with first vertex \( u \) and last vertex \( v \).

A walk (respectively trail) is closed if it has at least one edge and the first and last vertices of the walk (resp. trail) are the same. In our example \( v_0 = v_k \).

A graph is connected if each pair of vertices are joined by a walk. A component of a graph is a maximal (with respect to vertices) connected subgraph of \( G \). We can think of a component as an equivalence class of vertices where we have an equivalence relation that says that \( x \) is related to \( y \) if there is an \( x-y \)-walk in \( G \). For a connected graph \( G \), a cut edge is an edge \( e \) for which \( G \setminus e \) is disconnected.

A directed graph is strongly connected if for each pair of vertices \( x, y \) there is both a directed \( x-y \)-path as well as a directed \( y-x \)-path.

An eulerian circuit is a closed trail in which each edge of \( G \) is used. We typically allow a general graph in this problem.

A cycle is a closed trail in which the only pair of repeated vertices is the first and last vertices of the trail. Our definition of \( C_n \) refers to the isomorphism class of cycles of \( n \) edges. Note that a loop is a cycle and is of course a closed walk or trail. Also if we have two vertices \( x, y \) joined by an edge \( e \) then \( xeyex \) is a closed walk but not a cycle. If we have two edges \( e, f \) with endpoints \( x, y \) then we get a cycle \( xeyfx \). A chord of a cycle is an edge joining two vertices of the cycle not already joined by edges of the cycle.

A graph is bipartite if the vertices \( V(G) \) can be partitioned into \( X, Y (X \cup Y = V(G), X \cap Y = \emptyset) \) so that for each edge \( e \in E(G) \), one endpoint is in \( X \) and one endpoint is in \( Y \).

A subgraph \( H \) of \( G \) is a graph for which \( V(H) \subseteq V(G) \) and \( E(H) \subseteq E(G) \). Of course in order for \( H \) to be a graph, for each edge \( e \in E(H) \) we must have both endpoints in \( V(H) \).

An induced subgraph \( H = (V(H), E(H)) \) is a subgraph for which each edge \( e \) of \( G \) where both endpoints are in \( V(H) \) is also in \( E(H) \). We use the notation \( H = G[V(H)] \). We refer to a subgraph as induced even if we haven’t specified \( V(H) \) and in that case are asserting the existence of an appropriate \( V(H) \).

A subgraph \( H \) of \( G \) is called a spanning subgraph if \( V(H) = V(G) \). A spanning cycle is called a hamiltonian cycle.

A tree is a subgraph which is connected and has no cycles. We had alternate definitions in class. Note that a tree on more than one vertex must have a vertex of degree one.

A spanning tree is a spanning subgraph which is a tree and is also spanning.

A matching is a set \( M \subseteq E(G) \) of edges no two of which are incident. A perfect matching is a matching so for each vertex \( v \in V(G) \) there is an edge of \( M \) which is incident to \( v \). A spanning 1-regular subgraph is called a 1-factor and the edges form a perfect matching and this is then the same as a perfect matching. If we have a vector \( f = (f(v) : v \in V) \), then an \( f \)-factor is a spanning subgraph of \( G \) with degree at vertex \( v \) equal to \( f(v) \) for each \( v \in V \).
A clique is a set of vertices for which each pair are joined by edges; i.e. a set $S$ is a clique in $G$ if the subgraph induced by $S$ is $K_{|S|}$.

An independent set of vertices is a set $S$ of vertices for which no pair are joined by edges. Thus an independent set corresponds to a clique in $G^c$.

The Line Graph $L(G)$ is a graph obtained from $G$ with $L(G) = (E(G), E')$ where for $e_1, e_2 \in E(G)$, $e_1 e_2 \in E'$ if they have a vertex in common.

Graph parameters

1. The degree $d_G(v)$ (or just $d(v)$) of a vertex $v$ is the number of endpoints of edges equal to $v$, hence the number of incidences of edges with $v$ noting that we count a loop for two incidences. The degree sequence $d_1, d_2, \ldots$ of a graph (with $d_i = d_G(i)$), typically has $d_1 \geq d_2 \geq \cdots$.

2. A graph is cubic if every degree is 3. A graph is $r$-regular if every degree is $r$. If a vector $f = (f_1, f_2, \ldots, f_n)$ is given, then an $f$-factor is a subgraph $x = (x(e) : e \in E(G))$ of $G$ satisfying $x(e) \in \{0, 1\}$ with $d_x(i) = \sum e \text{ hits}_i x(e)$ being the associated degree. A fractional $f$-factor is a vector $x = (x(e) : e \in E(G))$ of $G$ satisfying $x(e) \in [0, 1]$ with $d_x(i) = \sum e \text{ hits}_i x(e)$.

3. We define $\delta(G) = \min_{v \in V} d(v)$, $\Delta(G) = \max_{v \in V} d(v)$

4. We have distance function $d_G(x, y) = d(x, y)$ being the length of shortest $x-y$-path

5. The diameter $\text{diam}(G) = \max_{x, y \in V} d(x, y)$.

6. $\kappa(G)$ is the (vertex) connectivity of $G$ and is the minimum number of vertices that must be deleted from $G$ to either disconnect the graph or leave a single vertex. A graph is $k$-connected if $\kappa(G) \geq k$. A cut in a graph is a set of vertices $S$ such that $G - S$ is disconnected.

7. $\kappa'(G)$ is the edge connectivity of $G$ and is the minimum number of edges that must be deleted from $G$ to disconnect the graph. A graph is $k$-edge-connected if $\kappa'(G) \geq k$. An edge cut in a graph is a set of edges of the form $[S, V \setminus S]$ where $S$ is a set of vertices $S \neq \emptyset, V(G)$. Then $G \setminus [S, V(G) \setminus S]$ is disconnected.

Special Graphs

1. $K_n$ denotes the simple graph on $n$ vertices with every pair of vertices joined by an edge. It is called the complete graph.

2. $C_n$ denotes the simple graph on $n$ vertices $\{1, 2, \ldots, n\}$ with $E(C_n) = \{12, 23, \ldots (n-1)n, n1\}$. It is called the cycle of length $n$.

3. $P_n$ denotes the simple graph on $n$ vertices $\{1, 2, \ldots, n\}$ with $E(P_n) = \{12, 23, \ldots (n-1)n\}$. It is called the path of length $n - 1$ since it has $n - 1$ edges.

4. $K_{r,s}$ denotes the simple graph on $r + s$ vertices where $|X| = r$ and $|Y| = s$ and for each choice $x \in X$ and $y \in Y$ we have $xy \in E(K_{r,s})$. It is called the complete bipartite graph on parts of size $r$ and $s$.

5. $Q_k$ is the $k$-dimensional hypercube consisting of $2^k$ vertices, each vertex corresponding to a different $(0,1)$-string of length $k$ and we join two vertices if their associated strings differ in exactly one position.
6. The *Petersen graph* (which has no special symbol) is the graph on 10 vertices with the property that each vertex has degree 3 and each pair of vertices is either joined by an edge or there is a path of length two (i.e. two edges) joining them.

7. $W_n$ is *wheel graph* on $n$ vertices with a central vertex joined to all others and $n - 1$ vertices whose induced subgraph is $C_{n-1}$.