1. Let $D = (N, A)$ be a directed graph with capacities on arcs denoted $u(e)$. A circulation is a flow $\mathbf{x} = (x(e) : e \in A)$ with

$$\sum_{e : t(e) = v} x(e) = \sum_{e : h(e) = v} x(e) \quad \forall v \in N$$

$$0 \leq x(e) \leq u(e) \quad \forall e \in A$$

A flow cycle is a flow $\mathbf{y}$ given by a directed cycle $C$ with a positive number $\epsilon(C)$ such that

$$y(e) = \begin{cases} \epsilon(C) & \text{if } e \in E(C) \\ 0 & \text{otherwise} \end{cases}$$

Of course $\epsilon(C) \leq \min_{e \in E(C)} u(e)$. Show that $\mathbf{x}$ can always be written as a sum of flow cycles.

2. Let $D = (N, A)$ be a directed graph with capacities on arcs denoted $u(e)$. A flow is $\mathbf{x} = (x(e) : e \in A)$ with

$$v(\mathbf{x}) = \sum_{e : t(e) = s} x(e) - \sum_{e : h(e) = s} x(e)$$

$$\sum_{e : t(e) = v} x(e) = \sum_{e : h(e) = v} x(e) \quad \forall v \in N \backslash \{s, t\}$$

$$0 \leq x(e) \leq u(e) \quad \forall e \in A$$

An s-t-flow path is a flow $\mathbf{y}$ given by a directed s-t-path $P$ and a positive number $\epsilon(P)$ such that

$$y(e) = \begin{cases} \epsilon(P) & \text{if } e \in E(P) \\ 0 & \text{otherwise} \end{cases}$$

Of course $\epsilon(P) \leq \min_{e \in E(P)} u(e)$. Show that $\mathbf{x}$ can always be written as a sum of s-t-flow paths and flow cycles.

3. Let $G$ be a $n$-vertex simple graph that decomposes into $k$ spanning trees. Given that also $\Delta(G) = \delta(G) + 1$, determine the degree sequence of $G$ in terms of $n$ and $k$.

4. Let $G$ be a tree where every vertex has degree 1 or degree $k$. Given $k$, for what values of $n(G)$ is this possible?

5. Prove that $K_{2m-1,2m}$ decomposes into $m$ spanning paths. (Can you make the problem more symmetric? Even then it is not trivial)

6. Consider a labeling of the edges of $K_n$ by integer labels such that no two incident edges share a label. Show that there is a trail of $n - 1$ edges consisting of edges $e_1, e_2, \ldots, e_{n-1}$ such that the labels are increasing (i.e. for $1 \leq i < j \leq n - 1$ the label of $e_i$ is less than the label of $e_j$. (this is a MUCH harder problem so don’t worry if you can’t solve it)