1. Let $G$ be a graph for which every pair of odd cycles has at least one vertex in common. Prove that $\chi(G) \leq 5$. Hint: if $G$ has a triangle, would you have a proof?

2. Given a graph $G$ with some $t$ vertices $X$ precoloured, show there is a monic polynomial $p(\lambda)$ in $\lambda$ of degree $|V(G)| - t$ such that $p_X(k)$ computes the number of legal colourings of $G$ with $k$ colours respecting the precoloured vertices $X$.

Now apply this to a sudoku puzzle. You can interpret a sudoku solution as a colouring of 81 vertices with 9 colours in an appropriately constructed graph where certain vertices $X$ have been precoloured. Give the associated graph on 81 vertices. Show that for a legitimate sudoku puzzle, $p_X(9) = 1$.

3. Show that for the Petersen graph $P$, we have $\chi'(P) = 4$.

4. Given an (undirected) connected graph $G = (V, E)$, we could ask for the minimum length odd cycle. Imagine we have an algorithm that finds a minimum total weight perfect matching (yes, polynomial algorithms exist). Consider the following graph $G_{xy}$ which you can envision as a copy of $G\setminus y$ on vertices $V\setminus y$ and a copy of $G\setminus x$ on vertices $V'\setminus x$ ($V'$ is just a copy of $V$) where we add the additional edges $zz'$ for $z \in V\setminus \{x,y\}$ to $E(G_{xy})$. Let $V' = \{v' : v \in V\}$:

$$G_{xy} = (V\setminus y) \cup (V'\setminus x'), E')$$

where $E' = \{(i, i') : i \in V\setminus \{x, y\}\} \cup \{(i, j) : (i, j) \in E, i, j \neq y\} \cup \{(i', j') : (i, j) \in E, i, j \neq x\}$

We give weights 0 to the edges $zz'$ and weights 1 to the remaining edges. Show that a minimum total weight perfect matching in $G_{xy}$ is equivalent to finding a path from $x$ to $y$ in $G$ that uses an even number of edges. Now use this to find a odd cycle of minimum length in $G$.

In fact we can solve the problem when the graph $G$ has positive edge weights and we seek a minimum total weight odd cycle. The weights in $G_{xy}$ are the same as in $G$ for corresponding edges and 0 weight for the edges $zz'$. Show that a minimum total weight perfect matching in $G_{xy}$ is equivalent to finding a minimum weight path from $x$ to $y$ in $G$ that uses an even number of edges. Now use this to find a odd cycle of minimum total weight in $G$.

(This will also extend to even cycles)