1. Let $G$ be a simple 3-regular graph.

a) Prove that if $G$ has a decomposition into $P_4$'s then $G$ has a perfect matching.

I asked this first since it can help you with b) or at least it did help me. A 3-regular graph must have $n(G)$ even since the sum of degrees must be even. We note that $G$ has $3n(G)/2$ edges and a perfect matching has $n(G)/2$ edges which is $1/3$ of the total. Suspicious! So if we have decomposed $G$ into $P_4$'s then we can take the middle edge of each $P_4$ (and hence have selected $1/3$ of the edges) and then note that the two endpoints of a middle edge of a $P_4$ have degree 2 in the $P_4$ and so no two middle edges will touch at a vertex since that vertex would be required to have degree 4 in $G$.

b) Prove that if $G$ has a perfect matching then $G$ has a decomposition into $P_4$'s.

We start by assuming that $G$ has a perfect matching $M$ and so $G – M$ is a 2-regular graph which then is a sum of cycles $C_1, C_2, \ldots, C_k$. Now for each edge $e$ of the matching we imagine by the above argument that we will extend $e$ at each end to form a $P_4$ by taking edges from the cycles. To do so in a consistent fashion orient each cycle (arbitrarily). Now if $e$ hits $C_i$ at vertex $v$ then choose the edge $vu$ where $vu$ is the next edge in $C_i$ following the order. Now since we have no loops we will never have difficulty pairing an endpoint $v$ of an edge $e$ of $M$ to an edge of the cycle containing $v$. When we are done we will have obtained $|M|$ $P_4$'s no two $P_4$'s sharing an edge. Then by a simple edge count we have used $3 \times n(G)/2$ edges and hence all the edges of $G$.

2. Let $G$ be a graph with all even degrees. Show that if it is true for some integer $\ell$ that $\kappa'(G) \geq 2\ell + 1$, then $\kappa'(G) \geq 2\ell + 2$.

Consider a set $E'$ such that $G\setminus E'$ is disconnected and $|E'| = \kappa'(G)$. Let $X$ be a component $G\setminus E'$. We claim that each edge of $E'$ has precisely one end in $V(X)$. If two ends of $e$ in $V(X)$ then $G\setminus (E' \cup e)$ is still disconnected with $X \setminus e$ disconnected from the rest of the graph. Similarly if no ends of $e$ are in $V(X)$, then $G\setminus (E' \cup e)$ is still disconnected with $X$ a component. These contradict the definition of $\kappa'(G)$. So our claim is verified.

Now we consider $G\setminus E'$ and note that in the component $X$, the sum of the degrees will be $\sum_{x \in X} d_G(x)$, which is even, minus $|E'|$. Since the sum of the degrees in $G[X]$ is even, we deduce that $|E'|$ is even. This gives the result.

3. The following problem explores a vector space of a connected graph $G$ consisting of spanning subgraphs of $G$ which have all even degrees (sometimes called even subgraphs). Addition in this vector space is modulo 2 sum (our symmetric difference) and scalar multiplication is over the field of 2 elements (the field formed by 0,1 with $1+1=0$ and all other operations as you would expect; one can think of 0 as ‘even’ and 1 as ‘odd’). Thus the spanning graph of no edges is the zero vector in this vector space. Select a spanning tree $T$ of $G$. We wish to show that

$$\mathcal{C} = \{C_e : e \in E(G) \setminus E(T), C_e \text{ is the unique cycle in } T + e\}$$

is a basis for the vector space $V$ (known as the cycle space).

a) Show that the cycles in $\mathcal{C}$ are linearly independent.

This follows easily since for each cycle $C_e$, the edge $e$ is not in any other cycle of $\mathcal{C}$. 

b) Show that if $C_1, C_2$ are cycles (not necessarily from $C$) with the property that $C_1 \setminus E(T) = C_2 \setminus E(T)$, then $C_1 = C_2$. (try symmetric difference)

If we take the vector space sum (which is the symmetric difference or the mod 2 sum) $C_1 + C_2$, we obtain a graph which has even degree at every vertex but whose edges lie in $E(T)$. But $T$ has no cycles and so no subgraphs with even degree at every vertex except for the subgraph of no edges. Thus $C_1 + C_2$ has no edges in $T$. But $C_1 \setminus E(T) = C_2 \setminus E(T)$ and so we conclude that $C_1 = C_2$.

c) Show that any cycle $C$ in $G$ is a unique linear combination of cycles in $C$. Here we are using the sum of subgraphs is modulo 2 sum (or our symmetric difference).

Let $E(C)$ consist of $A \cup B$ where $B \subseteq E(T)$. Then if we compute the subgraph $C' = \sum_{a \in A} C_a$

we have that as a sum of cycles it has even degree at every vertex and since $a \in C_a$ yet $a \notin C_d$ for any $d \in A \setminus a$, we deduce that $C' \setminus E(T) = A$. By b), we compare $C$ and $C'$ and deduce that $C = C'$. Thus $C$ is in the span of $C$ and the uniqueness of the coefficients in the linear expression follow from a).

d) Show that the dimension of the cycle space, namely the vector space consisting of all subgraphs of all even degrees, has dimension $e(G) - n(G) + 1$.

We note that c) asserts that the cycle space is the span of $C$ and a) says the cycles in $C$ are linearly independent. Thus dimension of the cycle space is $|C| = e(G) - e(T) = e(G) - n(G) + 1$.

4. Let $G$ be a bipartite cubic planar graph. Show that a planar drawing of $G$ has at least 6 faces of size 4 and give an example of such a graph.

Since $G$ is cubic, then $2|E| = 2|V|$. Since $G$ is bipartite, the smallest cycle sizes are 4,6. Assume $G$ has $k$ faces of size 4 and $\ell$ faces of size 6 or greater. Thus $k + \ell = |F|$. Also $2|E| \geq 4k + 6\ell$. We deduce that $2|E| - 6|F| \geq -2k$. Now multiplying Euler’s formula by 6, $12 = 6|V| - 6|E| + 6|F| = 6|V| - 4|E| - (2|E| - 6|F|) \leq 2k$.

Thus $k \geq 6$. We have equality if the faces are all of size 4 or 6.

5. Two variations on question 5 from practice midterm: We say that a spanning subgraph $H$ is a $\frac{a}{b}G$ (a fraction $\frac{a}{b}$ of $G$) if

$$d_H(v) = \frac{a}{b}d_G(v)$$

Solving that problem 5 yields that the edges of $G$ can be decomposed into two (spanning) $\frac{1}{2}G$ graphs.

a) Let $G$ be a connected graph with even degrees at all the vertices and an even number of edges. Show that the edges of $G$ can be decomposed into two $\frac{1}{2}G$ graphs.

Here we consider any component $C$ and the euler tour on a component. Take every second edge along the euler tour. For this to work we need that each component has an even number of edges. The result is a subgraph $H_C$ which has degree $d_{H_C}(v) = \frac{1}{2}d_C(v)$. Do this for each component. Then the union of such graphs is our desired graph $H$ and then the complement of $H$ in $G$ is another such graph $H^C$ such that $E(H) \cup E(H^C) = E(G)$. 
b) Let $G$ be a bipartite graph with all degrees a multiple of 3 and $\sum_{v \in V(G)} \frac{1}{3}d_G(v)$ is even. Show that the edges of $G$ can be decomposed into three $\frac{1}{3}G$ graphs.

For this we form a fractional subgraph $x$ with

$$x(e) = \frac{1}{3} \text{ so that } d_x(v) = \frac{1}{3}d_G(v).$$

We note that $G$ has the odd cycle property. Also the sum of the degrees we seek (our $f$ vector) is even. Hence there is a subgraph $H$ with $d_H(v) = \frac{1}{3}d_G(v)$ for all $v \in V(G)$. Now remove $H$ from $G$ to obtain a new graph $K$ with $d_K(v) = \frac{2}{3}d_G(v)$ for all $v \in V(G)$. Now repeat by decomposing $K$ into two $\frac{1}{2}K$ subgraphs noting that a $\frac{1}{2}K$ graph is in fact a $\frac{1}{3}G$ graph.

6. Consider an arbitrary drawing of $K_n$ in the plane. Necessarily it will have edge crossings (for $n \geq 5$). Prove that the numbers of pairs of edges that must cross is at least $\frac{1}{5}(\binom{n}{2})$. (note: $\frac{1}{5}(\binom{n}{2}) = \frac{1}{n-4}\binom{n}{5}$).

It is clear that any drawing of $K_5$ in the plane has at least one crossing pair of edges. Now consider a drawing of $K_n$ in the plane. Each 5 set of vertices (of which there are $\binom{n}{5}$) has a crossing pair of edges on 4 vertices. This crossing pair may be the crossing pair for up to $n-4$ 5-sets of vertices (each 5-set could come from adding one new vertex to the 4 vertices of the crossing pair). Thus the number of crossing pairs is at least $\frac{1}{n-4}\binom{n}{5}$.

7. Assume $G$ is a graph on $n$ vertices that is regular of degree $k$. Let $A$ be the $n \times n$ adjacency matrix. Let $I$ denote the $n \times n$ identity matrix and let $J$ denote the $n \times n$ matrix of all 1’s. Show that $A^2 + A = J + (k-1)I$ if an only if $G$ has diameter 2 and no 3-cycles or 4-cycles (girth at least 5). Compute $n$ as a function of $k$ in the case that $G$ has diameter 2 and no 3-cycles or 4-cycles.

We know that the $i,j$ entry of $A^2$ is the number of $i,j$-walks of two edges. Now $A^2 + A = J + (k-1)I$ yields that the $i,j$ entry of $A^2$ plus the $i,j$ entry of $A$ is either 1 if $i \neq j$ and $k$ if $i = j$. The case $i = j$ yields that the degree of every vertex is $k$. The case $i \neq j$ yields that either there is an edge joining $i$ and $j$ or there is a unique path of two edges joining $i$ and $j$ but not both. Immediately this gives that $G$ has diameter 2. Also it gives that there is no 3-cycle $i,j,k$ since then there is an edge joining $i$ and $j$ and a path $ikj$ of two edges joining $i$ and $j$, a contradiction. Similarly $G$ has no 4-cycle $ikjl$ since the 4-cycle yields two $i,j$- paths $ikj$ and $i\ell j$, a contradiction.

Assume $G$ has $A^2 + A = J + (k-1)I$. Then the graph is $k$-regular. Thus a given vertex $x$ is joined to $k$ vertices $N(x)$ and each of those vertices is joined to $k-1$ further vertices $(\bigcup_{v \in N(x)}(N(v) \setminus x))$ using the fact that $G$ has no 3-cycle of 4-cycle. This yields $1+k+k(k-1) = k^2 + 1$ vertices. Moreover this is all vertices of $G$ since $G$ has diameter 2 and so each vertex in $G$ is at distance at most 2 from $x$.

It is possible to use Linear Algebra to carry on and deduce that the only possible values for $k$ are 2,3,7,57. The associated graphs are called Moore Graphs.