1. Let $G$ be a $n$-vertex simple graph that decomposes into $k$ spanning trees. Given that also $\Delta(G) = \delta(G) + 1$, determine the degree sequence of $G$ in terms of $n$ and $k$.

By our hypothesis, $G$ has $k(n-1)$ edges (since a spanning tree has $n-1$ edges). The average degree $\bar{d}$ is $\frac{2k(n-1)}{n}$ which is very close to $2k$. In fact $2k - 1 \leq \bar{d} \leq 2k$ i.e. $\delta(G) = 2k - 1$ and $\Delta(G) = 2k$. The upper bound is easy. The lower bound follows using $k(n-1) \leq \binom{n}{2}$ (since the maximum number of edges for a graph on $n$ vertices is at most $\binom{n}{2}$). Then it follows that $k \leq \frac{n}{2}$ from which we may deduce that $2k - 1 \leq n - 1$ which, combined with $2k/n \leq 1$, yields $2k - 1 \leq \bar{d}$. We may compute that there will be $2k$ vertices of degree $2k - 1$ and $n - 2k$ vertices of degree $2k$. Note that $K_4$ can be decomposed into two paths of 3 edges which corresponds to $k = 2$ and $n = 4$ and so $K_4$ has 4 vertices of degree 3 and 0 vertices of degree 4.

2. Given a tree $T$ on at least two vertices, we are often interested in the number of vertices of degree 1 (called leaves). Show that

\[ \sum_{v} d_T(v) = 2 + \sum_{v \text{ with } d_T(v) \geq 3} (d_T(v) - 2). \]

One approach is to use induction on the number of vertices. Let $T$ have $n$ vertices Let $u$ have $d_T(u) = 1$ and let $e = (u, v)$ be the incident edge. Now consider $T' = T - e$. We deduce that $T'$ is a forest consisting of two trees, one a tree on 1 vertex $u$ and a second tree $T''$ on the remaining vertices $V(T) \setminus \{u, v\}$. We have $d_{T'}(v) = d_T(v) - 1$ and $d_{T'}(w) = d_T(w)$ for all $w \in V(T) \setminus \{u, v\}$. Apply induction to $T'$. The result is true for a tree on 2 vertices, both of which must have degree 1.

Alternately use the equation

\[ \sum_v d_T(v) = 2|E(T)| - 2|V| - 2. \]

\[ \sum_v (d_T(v) - 2) = -2 \]

\[ \sum_{v : d_T(v) \geq 3} (d_T(v) - 2) - \sum_{v : d_T(v) = 1} 1 = -2 \]

from which we deduce our equation after rearranging.

3. Let $D = (N, A)$ be a directed graph with capacities on arcs denoted $c(e)$. A circulation is a flow $x = (x(e) : e \in A)$ with

\[ \sum_{e : t(e) = v} x(e) = \sum_{e : h(e) = v} x(e) \quad \forall v \in N \]

\[ 0 \leq x(e) \leq c(e) \quad \forall e \in A \]

Show that $x$ can be written as a sum of flow cycles.

Let $m = |\{e \in A : x(e) > 0\}|$. We show by induction on $m$ that $x$ can be written as a sum of at most $m$ flow cycles. We find an edge $e = uv$ with $x(e) > 0$ and then use our $\rho$
5. Prove that $K_4$ decomposes into $m$ spanning paths. (this is a little difficult)

Let the 4 vertices $1, 2, \ldots, 2m$ and $1', 2', \ldots, 2m'$. We can get our first cycle

$$(1, 1'), (1, 2'), (2, 2'), (2, 3'), (3, 3'), (3, 4'), \ldots$$

by taking the edges $(i, i'), (i, (i+1)')$. We can get our second cycle by cycling the second indices by 2, namely the edges

$$(1, 3'), (1, 4'), (2, 4'), (2, 5'), (3, 5'), (3, 6'), \ldots$$

the edges $(i, (i + 2)'), (i, (i + 3)')$. Thus the $k$th cycle will have edges of the form $(i, (i + 2k - 2)'), (i, (i + 2k - 1)')$. We are imagining all node labels modulo $2m$ to take advantage of the symmetry.

Now if we remove a vertex, our cycles become paths and our graph becomes $K_{2m-1, 2m}$.

6. Consider a labeling of the edges of $K_n$ by integer labels such that no two incident edges share a label. Show that there is a trail (i.e. walk with no repeated edges) of $n-1$ edges consisting of edges $e_1, e_2, \ldots, e_{n-1}$ such that the labels are increasing (i.e. for $1 \leq i < j \leq n-1$ the label of $e_i$ is less than the label of $e_j$. (this is a MUCH harder problem so don’t worry if you can’t solve it)

I was told this solution by Peter Winkler. He pointed out a way to create $n$ trails of the desired increasing type which in total use every edge twice i.e. the sum of the lengths of all $n$ paths is $2 \binom{n}{2} = n(n - 1)$. We deduce that some increasing trail is of length at least $n - 1$. 

Argument growing a directed path from $u$ first using $e = uv$ such that each edge $f$ of the directed path has $x(f) > 0$. We use that if there is an arc into a node with non-zero flow entering the node then by the requirements of a circulation there must be some arc out of the node with non-zero flow. Eventually we find a directed cycle which yields a flow cycle $C$ by taking $e(C) = \min_{e \in C} x(e)$. Now subtract the flow around the flow cycle from the circulation $x$. We obtain a new circulation with $< m$ arcs having non-zero flow. Thus by induction $x$ is a sum of at most $m$ flow cycles.

Note that the induction on the number of arcs with non-zero flow gives us the decomposition after at most $|A|$ cycles. Removing flow cycles arbitrarily without this idea (this would be unlikely!) might not terminate.

4. Let $D = (N, A)$ be a directed graph with capacities on arcs denoted $c(e)$. A flow is $x = (x(e) : e \in A)$ with

$$v(x) = \sum_{e : t(e) = s} x(e) - \sum_{e : h(e) = s} x(e)$$

$$\sum_{e : t(e) = v} x(e) - \sum_{e : h(e) = v} x(e) \quad \forall v \in N \setminus \{s, t\}$$

$$0 \leq x(e) \leq c(e) \quad \forall e \in A$$

Show that $x$ can be written as a sum of $s$-$t$-flow paths and flow cycles.

We use question above. Take the given flow and digraph and add an arc $t \to s$ with $x(t, s) = v(x)$, the net flow out of $s$. The result is a circulation for which, by the above question, the new flow $x$ is a sum of flow cycles. Those flow cycles using $t \to s$ become $s$-$t$-flow paths when you delete $t \to s$. Those flow cycles not using $t \to s$ are flow cycles in $D$. 


The idea is to start \( n \) paths from each vertex. At each step in our algorithm each vertex will be the end of some increasing trail. Go through the labels in increasing order. When you examine label \( i \), for each edge \( e = xy \) with label \( i \), extend the path from \( x \) along \( e \) to \( y \) and extend the path from \( y \) along \( e \) to \( x \). Given that the edges of label \( i \) are not incident, this process is not ambiguous. Each edge of label \( i \) has been incorporated in two increasing trails. At the end of examining edges of label \( i \) we again have \( n \) increasing trails terminating at the \( n \) vertices and so we may repeat until all labels have been examined. Each edge will appear in two increasing trails. This is what we were looking for above.