1. (10 marks) Let $C_1, C_2$ be two different cycles of $G$ with $E(C_1) \cap E(C_2) \neq \emptyset$. Show that there is a third cycle $C_3$ in $G$.

2. (10 marks) Let $G$ be a simple graph on 34 vertices with 5 vertices of degree 3 and 9 vertices of degree 1 and 20 vertices of degree 2. Show that $G$ is not connected.

3. (10 marks)
   a) Let $G$ be a 4-regular connected graph. Show that $G$ has no cut edge (A cut edge is an edge $e$ for which $G \setminus e$ is disconnected).
   
   b) Let $G$ be a connected cubic graph and let $G$ have an cut edge $e = xy$. Show that any perfect matching of $G$ (if one exists) must use the edge $e$.

4. (10 marks)
   a) Show that there is no cubic plane graph with exactly one face of size 7 and the rest of the faces have sizes 4 or 6.
   
   b) Let $G$ be a connected plane graph with all faces of size at least 4. Show that $e(G) \leq 2n(G) - 4$.
   
   c) Let $G$ be a simple connected plane graph with $s$ faces of size 5 and $t$ faces of size 6 and no other faces. Assume $G$ is cubic (all degrees are 3). Determine $s$.

5. (10 marks)
   a) Let $G$ be a $d$-regular graph with $\kappa'(G) \geq d$. Show that for each $k$ (with $k \geq 1$) and each $S \subseteq V(G)$ with $|S| = k$, show that $G - S$ has at most $k$ components.
   
   b) Show that if $G$ is $d$-regular with $\kappa'(G) \geq d$ and $|V(G)|$ being even, then $G$ has a perfect matching.

6. (10 marks) Assume the directed graph $D = (N, A)$ is strongly connected. Show that there is a directed spanning subgraph $D' = (N, A')$ with $A' \subseteq A$ satisfying that $D'$ is strongly connected and $|A'| \leq 2|N|$.
7. (10 marks) Let $T$ and $T'$ be two edge disjoint spanning trees on the same set of $n$ vertices. Let $G$ be the union of $T$ and $T'$, namely $G$ is the graph with $V(G) = V(T) = V(T')$ and $E(G) = E(T) \cup E(T')$.

a) Show that $\kappa'(G) \geq 2$.

b) Show that $\kappa'(G) < 4$.

c) Choose a value for $n$ and choose two spanning trees $T$, $T'$ whose union $G$ has $\kappa'(G) = 3$.

8. (10 marks) Let $W$ be a closed walk in a simple graph $G$. Let $H$ be the spanning subgraph of $G$ consisting of one copy of each edge that was used an odd number of times in $W$. Prove that for each $v \in V(G)$, $d_H(v)$ is even.

9. (10 marks) Let $G$ be a planar graph which is uniquely 4-colourable, namely every 4-colouring of the vertices is the same up to a renaming of the colours. Show that $e(G) = 3n(G) - 6$.

10. (10 marks) Let $D$ be an orientation of the complete graph $K_n$. Assume that $D$ is strongly connected. Thus $D$ has a directed cycle. Show that $D$ has a directed cycle using all the vertices (a spanning directed cycle).

total 100 marks
1. Let $G$ be a connected graph with $2k$ vertices of odd degree and the rest have even degree. Show that $G$ has a trail with at least $|E(G)|/k$ edges.

2. (10 marks) Show how any simple graph $G$ can be decomposed into even closed trails (i.e. closed trails using an even number of vertices), paths whose endpoints are odd degree vertices in $G$ and vertex disjoint odd cycles.

3. (10 marks) Let $W$ be a closed walk in a simple graph $G$. Let $H$ be the spanning subgraph of $G$ consisting of those edges used an odd number of times in $W$. Prove that for each $v \in V(G)$, $d_H(v)$ is even. (Note: the spanning subgraph $H$ could have no edges).

4. (10 marks) An undirected graph $G$ is orientable if the edges can be oriented so the the resulting directed graph $D(G)$ is strongly connected. e.g. $C_n$ is orientable.
   a) Assume $G$ is a connected graph with a cut edge $e = xy$. Show that $G$ is not orientable.
   b) Assume that $G$ is a 2-edge connected graph on at least 4 vertices, then $G$ is orientable.

5. (10 marks) Let $G$ be a simple graph with $\Delta(G) = 2$ and $G$ has 4 vertices of degree 1. Show that $G$ is disconnected.

6. (10 marks) Let $G$ be a graph with $k$ components and $e(G) = n(G) - k$. Show that $G$ has no cycles.

7. (10 marks)
   a) Let $G$ be a simple graph with $\chi(G) = 3$. Show that there is a subset $S$ of the vertices with $|S| \geq (2/3)|V(G)|$ such that the subgraph of $G$ induced by the vertices of $S$ is bipartite.
   b) Extend this to graphs with $\chi(G) = k$ and determine a large fraction of the vertices which induce a bipartite subgraph of $G$. Show the result is best possible for $G = K_n$.

8. (10 marks) Let $G$ be a cubic simple graph. Assume that $G$ has a Hamiltonian cycle. Show that we can decompose $E(G)$ into 3 perfect matchings. Does the same result hold if we merely assume $G$ has a spanning subgraph regular of degree 2 (a 2-factor)?

9. (10 marks) Let $G$ be a connected simple graph. Let $G$ have a cut vertex $v$ so that $G$ can be thought of as the union of two connected subgraphs $H, K$ which overlap on the single vertex $v$. Show that
   \[
   \chi(G; k) = \frac{1}{k} \chi(H; k) \chi(K; k).
   \]