MATH 443 Various Notations

In this course, a graph G consists of a finite set $V(G) = V$ of vertices and a finite set $E(G) = E$ of edges such that each edge has associated with it two endpoints which need not be distinct. A *loop* is an edge with both endpoints the same e.g. $e = vv$.

A graph G is called *simple* if it has no multiple edges or loops.

A *walk* of *length* k is a sequence

$$W \qquad v_0e_1v_1e_2v_2 \cdots e_kv_k$$

such that $v_i \in V(G)$, $e_i \in E(G)$ and the endpoinst of e_i are v_{i-1},v_i or in other words $e_i = v_{i-1}v_i$. We say that W is a v_0-v_k-walk

A *trail* is a walk with no repeated edges.

A *path* is a walk with no repeated vertices.

A u-v-walk (respectively trail or path) is a walk (respectively trail or path) with first vertex u and last vertex v.

A walk (respectively trail) is *closed* if it has at least one edge and the first and last vertices of the walk (resp. trail) are the same. In our example $v_0 = v_k$.

An *eulerian circuit* is a closed trail in which each edge of G is used.

A graph is *connected* if each pair of vertices are joined by a walk. A *component* of a graph is a maximal (with respect to vertices) connected subgraph of G. We can think of a component as an equivalence class of vertices where we have an equivalence relation that says that x is related to y if there is an x-y-walk in G.

A *cycle* is a closed trail in which the only pair of repeated vertices is the first and last vertices of the trail. Our definition of C_n refers to the isomorphism class of cycles of n edges. Note that a loop is a cycle and is of course a closed walk or trail. Also if we have two vertices x, y joined by an edge e then $xeyex$ is a closed walk but not a cycle. If we have two edges e, f with endpoints x, y then we get a cycle $xeyfx$.

A graph is *bipartite* if the vertices $V(G)$ can be partitioned into X, Y ($X \cup Y = V(G)$, $X \cap Y = \emptyset$) so that for each edge $e \in E(G)$, one endpoint is in X and one endpoint is in Y.

A *subgraph* H of G is a graph for which $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. Of course in order for H to be a graph, for each edge $e \in E(H)$ we must have both endpoints in $V(H)$.

An *induced subgraph* $H = (V(H), E(H))$ is a subgraph for which each edge e of G where both endpoints are in $V(H)$ is also in $E(H)$. We refer to a subgraph as induced even if we haven't specified $V(H)$ and in that case are asserting the existence of an appropriate $V(H)$.

A subgraph H of G is called a *spanning subgraph* if $V(H) = V(G)$.

A *tree* is a subgraph which is connected and has no cycles. We had alternate definitions in class. Note that a tree on more than one vertex must have a vertex of degree one.

A *spanning tree* is a subgraph which is a tree and is also spanning.

A *matching* is a set $M \subseteq E(G)$ of edges no two of which are incident. A *perfect matching* is a matching so for each vertex $v \in V(G)$ there is an edge of M which is incident to v. The book distinguishes this from a *1-factor* which is a spanning subgraph of G where each vertex has degree 1. If we have a vector $f = (f(v) : v \in V)$, then an f-factor is a spanning subgraph of G with degree at vertex v equal to $f(v)$ for each $v \in V$.

A *clique* is a set of vertices for which each pair are joined by edges; i.e. a set S is a cligue in G if the subgraph induced by S is $K_{|S|}$.

An *independent set* of vertices is a set S of vertices for which no pair are joined by edges. Thus an independent set corresponds to a clique in G^c.

Graph parameters

1. $e(G) = |E(G)|$, $n(G) = |V(G)$. Sometimes $n(G)$ is called the order of G.
2. The *degree* of a vertex v is the number of endpoints of edges equal to v, hende the number of incidences of edges with v noting that we count a loop for two incidences.
3. We define
$$\delta(G) = \min_{v \in V} d(v), \qquad \Delta(G) = \max_{v \in V} d(v)$$
4. The *girth* of a graph is the length of the shortest cycle in a graph.
5. $\tau(G)$ is the number of (labeled) spanning trees of G and satisfies the recurrence $\tau(G) = \tau(G - e) + \tau(G \cdot e)$. Here $G \cdot e$ denotes the graph obtained from G by contracting e.
6. $\kappa(G)$ is the (vertex) connectivity of G and is the minimum number of vertices that must be deleted from G to either disconnect the graph or leave a single vertex. A graph is k-connected if $\kappa(G) \geq k$. A *cut* in a graph is a set of vertices S such that $G - S$ is disconnected.
7. $\kappa'(G)$ is the edge connectivity of G and is the minimum number of edges that must be deleted from G to disconnect the graph. A graph is k-edge-connected if $\kappa'(G) \geq k$. An *edge cut* in a graph is a set of edges of the form $[S, V \backslash S]$ where S is a set of vertices $S \neq \emptyset, V(G)$. Then $G \backslash [S, V(G) \backslash S]$ is disconnected.
8. $\omega(G)$ is the size of the largest clique (largest complete subgraph) in G
9. $\alpha(G)$ is the size of the largest indepenedent set in G (largest complete subgraph in G^c)
10. $\chi(G)$ is the minimum number of colours required to give a proper colouring to G.
11. $\chi(G; k)$ is the chromatic polymomial in k of G that evaluates to the number of proper colourings of G using at most k colours. It satisfies the recurrence $\chi(G; k) = \chi(G - e; k) - \chi(G \cdot e; k)$.
12. $a(G)$ is the number of acyclic orientations of G. It satisfies the recurrence $a(G) = a(G - e) + a(G \cdot e)$.
13. A *Hamiltonian* path is a path of G of length $n(G) - 1$.
14. A *Hamiltonian* circuit is a cycle of G of length $n(G)$.

Special Graphs

1. K_n denotes the simple graph on n vertices with every pair of vertices joined by an edge. It is called the *complete* graph.
2. C_n denotes the simple graph on n vertices $\{1, 2, \ldots, n\}$ so that $E(C_n)$ consists of $12, 23, \ldots (n-1)n, n1$. It is called the *cycle* of length n.
3. P_n denotes the simple graph on n vertices $\{1, 2, \ldots, n\}$ so that $E(P_n)$ consists of $12, 23, \ldots (n-1)n$. It is called the *path* of length $n - 1$ since it has $n - 1$ edges.
4. $K_{r,s}$ denotes the simple graph on $r + s$ vertices where $|X| = r$ and $|Y| = s$ and for each choice $x \in X$ and $y \in Y$ we have $xy \in E(K_{r,s})$. It is called the *complete bipartite* graph on parts of size r and s.

5. Q_k is the k-dimensional hypercube consisting of 2^k vertices, each vertex corresponding to a different (0,1)-string of length k and we join two vertices if their associated strings differ in exactly one position.

6. The *Petersen graph* (which has no special symbol) is the graph on 10 vertices with the property that each vertex has degree 3 and each pair of vertices is either joined by an edge or there is a path of length two (i.e. two edges) joining them.

7. A graph is *chordal* if every cycle of length at least 4 has a chord.