MATH 443 Assignment #6 Due Wednesday Nov 25.

1. Consider the following operation on a cubic graph G. For each vertex v of G, replace the vertex and its three incident edges e_1, e_2, e_3 by a 'copy' of the Petersen graph, namely a copy of the Petersen graph where one vertex of the Petersen graph is deleted but we keep the three incident edges f_1, f_2, f_3 and identify edge e_1 with f_1, identify e_2 with f_2 and e_3 with f_3. If G has $n(G)$ vertices, then the new graph G' has $n(G') = 9n(G)$ since each vertex of G is replaced by 9 vertices. The new graph is cubic. Now if $c(G)$ denotes the longest cycle in G (and $c(G) \geq 9$), then show that $c(G') \leq 8c(G)$.

   We can repeat this process. Use this to show that there are cubic graphs where $c(n)$ is small compared to $n(G)$, i.e. where $c(G) \leq n(G)^p$ where p is less than 1.

2. Find a simple planar graph that has precisely 2 vertices of degree 6 and all the rest have degree 5. Icosahedron?

3. Show that a simple plane graph with no faces of size 3 or less must have at most $n(G) - 2$ faces. (for example the cube Q_3).

4. Let G be a graph with the property that every pair of odd cycles have at least one vertex in common. Show that $\chi(G) \leq 5$. You might first note that you would be done if the smallest odd cycle is C_3.

5. Consider an arbitrary drawing of K_n in the plane with crossings (for $n \geq 5$). Prove that the numbers of pairs of edges that must cross is at least $\frac{1}{5}\binom{n}{4}$. Hmm. $\frac{1}{5}\binom{n}{4} = \frac{1}{n-4}\binom{n}{5}$.

6. Show that a bipartite cubic simple plane graph G can have its faces coloured with only 3 colours. Let G^* be the planar dual. Our goal is to show that $\chi(G^*) \leq 3$ to establish our claim. Let me suggest the following steps.

   a) Show that the faces of G^* can be coloured with just two colours.

   b) What are the face sizes for G^*? Orient the faces of one colour all clockwise. Then the edges of the faces of the other colour are oriented counterclockwise. Now show that for each cycle in G^* that

   $$\text{the no. of forward arcs in } C - \text{ the number of backward arcs in } C \equiv 0 (\text{mod } 3)$$

   c) Now choose a single vertex $x \in V(G^*)$ and let V_i denote the vertices joined to x by paths where the number of forward arcs minus the number of backwards arcs is $\equiv i (\text{mod } 3)$. Verify that this partitions $V(G^*)$ into three sets V_0, V_1, V_2, each of which is an independent set. This should complete the proof.