MATH 443 Assignment #5 Due Monday Nov 16.

1. Let G be a simple graph with $\chi(L(G)) = 2$. Show that G is a vertex disjoint union of paths and even cycles.

2. Let G be a simple graph on n vertices. Let G^c denote the complement of G; namely $V(G^c) = V(G), E(G^c) = E(K_n)\backslash E(G)$. Show that

$$\chi(G) + \chi(G^c) \leq n + 1$$

   (try induction on n)

3. Let G be a graph of girth at least 6 with $\chi(G) = k$ and $n(G) = n$. Form a new graph H consisting of $\binom{kn}{n}$ vertex disjoint copies of G each indexed by a unique element of $\binom{[kn]}{n}$ and a set S of kn new vertices. For each subset of S of size n, join it to the corresponding copy of G by a matching ($|S| = n = |V(G)|$). Show that the resulting graph has girth 6 and chromatic number at least $k+1$. Hint: If H has $\chi(H) = k$, then there is a subset of S of size n all of whose vertices receive the same colour. Consider the corresponding copy of G. When you are done the construction you have a construction for each k of a graph G with girth 6 and $\chi(G) \geq k$.

4. The following problem explores a vector space of a connected graph G consisting of spanning subgraphs of G which have all even degrees (sometimes called *even subgraphs*). Addition in this vector space is modulo 2 sum (our symmetric difference) and scalar multiplication is over the field of 2 elements (the field formed by 0,1 with 1+1=0 and all other operations as you would expect; one can think of 0 as 'even' and 1 as 'odd'). Thus the spanning graph of no edges is the zero vector in this vector space. Select a spanning tree T of G. We wish to show that

$$\mathcal{C} = \{C_e : e \in E(G)\backslash E(T), C_e \text{ is the unique cycle in } T + e\}$$

   is a basis for the vector space V (known as the cycle space).

   a) Show that the cycles in $\mathcal{C}$ are linearly independent.

   b) Show that if C_1, C_2 are cycles (not necessarily from $\mathcal{C}$) with the property that $C_1\backslash E(T) = C_2\backslash E(T)$, then $C_1 = C_2$. (try symmetric difference)

   c) Show that any cycle C in G is a unique linear combination of cycles in $\mathcal{C}$. Here we are using the sum of subgraphs is modulo 2 sum or our symmetric difference.

   d) Show that the dimension of the cycle space, namely the vector space consisting of all subgraphs of all even degrees, has dimension $e(G) - n(G) + 1$.

5. Let G be a simple graph. Let $m(G)$ be the length of the longest path in G. Show that

$$\chi(G) \leq m(G) + 1$$

   (Hint: this is tricky. Consider a maximal independent subset of the endpoints of longest paths and delete it.)