

Consider the following (primal) LP

$$\begin{aligned} \max \quad & \mathbf{c} \cdot \mathbf{x} \\ & A\mathbf{x} \leq \mathbf{b}. \\ & \mathbf{x} \geq \mathbf{0} \end{aligned}$$

1. Fundamental Theorem of Linear Programming.

A linear program satisfies exactly one of the following:

- i) The LP is infeasible (i.e. has no feasible solution).
- ii) The LP has an optimal solution.
- iii) The LP is unbounded (i.e. (for a maximization problem) for any bound L , there exists a feasible solution \mathbf{x} to the LP with $\mathbf{c} \cdot \mathbf{x} > L$)

The following fact has been added by Chvátal:

- iv) If there is a feasible solution then there is a basic feasible solution and if there is an optimal solution then there is a basic feasible solution which is optimal.

In comparison, iv) is little used in this course. Although the simplex algorithm only looks at basic feasible solutions, in the end we usually are not unhappy looking at any solution even if it is not a basic feasible solution. When I ask you to use the Fundamental Theorem of LP, I will be referring to i),ii),iii). The statement iv) follows from the proof of i) (Phase One) and ii) (Phase Two).

The dual LP of the primal given above is

$$\begin{aligned} \min \quad & \mathbf{b} \cdot \mathbf{y} \\ & A^T \mathbf{y} \geq \mathbf{c}. \\ & \mathbf{y} \geq \mathbf{0} \end{aligned}$$

2. Weak Duality Theorem. Let \mathbf{x} be a feasible solution to the primal and let \mathbf{y} be a feasible solution to the dual. Then

$$\mathbf{c} \cdot \mathbf{x} \leq \mathbf{b} \cdot \mathbf{y}.$$

Note we often use the weak duality theorem to assert optimality if we have found feasible solutions to the primal and dual that agree on their objective function values.

3. Strong Duality Theorem. If either

- i) the primal has an optimal solution or the dual has an optimal solution
- or

- ii) there exists feasible solutions to both the primal and the dual

then there exists an optimal solution \mathbf{x}^* to the primal and an optimal solution \mathbf{y}^* to the dual with

$$\mathbf{c} \cdot \mathbf{x}^* = \mathbf{b} \cdot \mathbf{y}^*.$$

Note that Weak Duality combined with the Fundamental Theorem of Linear Programming will mean that if there exists feasible solutions to both the primal and the dual then we deduce the primal has an optimal solution.

4. Complementary Slackness Theorem.

Let \mathbf{x} be a feasible solution to the primal and let \mathbf{y} be a feasible solution to the dual. Then \mathbf{x} and \mathbf{y} are optimal to their respective LP's if and only if Complementary Slackness holds, namely

$$i\text{th primal slack} \cdot y_i = 0 \text{ for } i = 1, 2, \dots, m,$$

$$x_j \cdot j\text{th dual slack} = 0 \text{ for } j = 1, 2, \dots, n.$$

Alternatively, we can express complementary slackness as:

$$i\text{th primal slack} = 0 \text{ or } y_i = 0 \text{ or both for } i = 1, 2, \dots, m,$$

$$x_j = 0 \text{ or } j\text{th dual slack} = 0 \text{ or both for } j = 1, 2, \dots, n.$$

5. Economic Interpretation of the Dual Variables.

Assume we have an optimal solution \mathbf{x}^* to our primal and $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_m^*)^T$ to our dual. If we change the right hand side of the i th inequality by Δ then (if we are lucky; this is only expected to be valid for Δ small) we will change the optimal value of the objective function ($\mathbf{c} \cdot \mathbf{x}^*$) by $y_i^* \cdot \Delta$. A condition on Δ , so that the predicted change in the objective function value is correct, is that the current basis continues to yield a feasible solution even when the right hand side of the i th inequality is changed by Δ . In the language of the Revised Simplex Formulas, $B^{-1}(\mathbf{b} + \mathbf{b}') \geq \mathbf{0}$ where \mathbf{b}' is the vector with a Δ in the i th place and zeros elsewhere.

The dual variables are often called 'shadow prices' or 'marginal values' in view of the above result. We need an economic interpretation of the LP where the objective function measures profit. Assume the variable x_j represent the amount of product j that is to be produced. The i th inequality takes the form $\sum_{j=1}^n a_{ij}x_j \leq b_i$ and so we may interpret a_{ij} as the amount of resource i required by a unit of product j and b_i is the total availability of resource i . Thus changes to b_i by Δ can be interpreted as increasing the availability of resource i by Δ . If we know the objective function changes by $y_i^* \cdot \Delta$, then the marginal value (how much profit you can make by having more of resource i) of a unit of resource i is y_i^* , the i th dual variable. That is to say changing the availability of resource i by 1 changes the objective function value by y_i^* (assuming $\Delta = 1$ is a lucky value suitably small with $B^{-1}(\mathbf{b} + \mathbf{b}') \geq \mathbf{0}$). These marginal values are typically valid in a small limited range. In the sensitivity analysis part of the course we will quantify this but you can get the information from LINDO as well.

Theorem for marginal values Let $\mathbf{b}' = (\Delta_1, \Delta_2, \dots, \Delta_m)^T$ and consider

$$\begin{array}{ll} \max & \mathbf{c} \cdot \mathbf{x} \\ \text{primal:} & \mathbf{Ax} \leq \mathbf{b}' \\ & \mathbf{x} \geq \mathbf{0} \end{array} \quad \begin{array}{ll} \max & \mathbf{c} \cdot \mathbf{x} \\ \text{altered primal:} & \mathbf{Ax} \leq \mathbf{b} + \mathbf{b}' \\ & \mathbf{x} \geq \mathbf{0} \end{array}$$

where $\mathbf{y}^* = (y_1^*, y_2^*, \dots, y_m^*)^T$ is a optimal solution to the dual (of the primal LP) and the optimal value for the primal objective function is z^* . Then the optimal value of the objective function for the altered primal is $\leq z^* + (\sum_{i=1}^m y_i^* \cdot \Delta_i)$ with equality holding for restricted (small) values of \mathbf{b}' . For certain \mathbf{b}' , the altered primal may be infeasible.