In general, our initial dictionary for an LP with slack variables, may not yield a feasible solution.

Maximize \(-x_1 + 3x_2 + x_3 + x_4\)
\[
\begin{align*}
2x_1 + x_2 - x_4 & \leq 4, \quad x_1, x_2, x_3 \geq 0 \\
-2x_1 + x_3 + x_4 & \leq -2 \\
2x_2 + 2x_3 & \leq 3
\end{align*}
\]

We now add slack variables
\[
x_5 = 4 - 2x_1 - x_2 + x_4 \\
x_6 = -2 + 2x_1 - x_3 - x_4 \\
x_7 = 3 - 2x_2 - 2x_3 + x_4 \\
z = -x_1 + 3x_2 + x_3 + x_4
\]

A basic solution associated with the dictionary is obtained by setting the non basic variable equal to 0
\[
x_1 = x_2 = x_3 = x_4 = 0 \quad \text{yielding} \quad x_5 = 4, \quad x_6 = -2 \quad \text{and} \quad x_7 = 3. \quad \text{This is not feasible since} \quad x_6 < 0.
\]

So how do we proceed? We add an artificial variable to achieve feasibility and then attempt to drive the artificial variable to 0 using our simple method. This is considered phase one of the two phase method.

We use the notation \(x_0\) for the artificial variable partly so that when applying ‘Anstee’s rule’ for choosing a leaving variable that \(x_0\) would be preferred over other choices that tie \(x_0\). We use the objective function maximize \(w = -x_0\) to drive \(x_0\) to 0.

Now at this point you might say you are done since the coefficients in the \(w\) row are all negative but of course we haven’t reached feasibility. The idea is we can to a special pivot to feasibility that results is a feasible dictionary but not optimal (in terms of minimizing \(w = -x_0\)).

We choose \(x_0\) to enter the basis and choose the leaving variable so we achieve feasibility
\[
x_5 = 4 + x_0 \geq 0 \quad \text{so} \quad x_0 \geq -4 \\
x_6 = -2 + x_0 \geq 0 \quad \text{so} \quad x_0 \geq 2 \\
x_7 = 3 + x_0 \geq 0 \quad \text{so} \quad x_0 \geq -3
\]

We must choose \(x_0\) to increase to 2 driving \(x_6\) to 0 and so \(x_0\) enters the basis and \(x_6\) leaves the basis.

Now after this special pivot to feasibility, we can proceed as before using the simplex method to minimize \(w = -x_0\) and hence drive \(x_0\) to zero if possible. Note that the clever choice of \(x_0\) means
that if $x_0$ is driven to zero in a pivot then it will be chosen to leave the basis (using Anstee’s Rule) and hence we can say goodbye to $x_0$ at this point.

For our particular dictionary above, we choose $x_1$ to enter and then $x_0$ leaves (oddly quick!)

$$
x_5 = 2 + 2x_0 - x_2 - x_3 - x_6
x_1 = 1 - \frac{1}{2}x_0 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6
x_7 = 3 + x_0 - 2x_2 - 2x_3
w = -x_0
$$

We can now delete $x_0$ and $w$ since they are no longer needed:

$$
x_5 = 2 - x_2 - x_3 - x_6
x_1 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6
x_7 = 3 - 2x_2 - 2x_3
$$

We do need $z$ and then to minimize it.

$$
z = -x_1 + 3x_2 + x_3 + x_4
$$

This is not so good since $x_1$ is in the basis of our dictionary so we substitute to eliminate $x_1$ from $z$:

$$
z = -(1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6) + 3x_2 + x_3 + x_4
z = -1 + 3x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6
$$

This yields the dictionary

$$
x_5 = 2 - x_2 - x_3 - x_6
x_1 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6
x_7 = 3 - 2x_2 - 2x_3
z = -1 + 3x_2 + \frac{1}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6
$$

We are now ready to proceed as before to maximize $z$. The computation of the new $z$ row and the subsequent pivots are considered the second phase of the two phase method.

So by our standard pivot process, we choose $x_2$ to enter and $x_7$ to leave.

$$
x_5 = \frac{1}{2} + \frac{1}{2}x_7
x_1 = 1 + \frac{1}{2}x_3 + \frac{1}{2}x_4 + \frac{1}{2}x_6
x_2 = 3 - \frac{1}{2}x_7 - x_3
z = -\frac{7}{2} + \frac{1}{2}x_7 - \frac{5}{2}x_3 + \frac{1}{2}x_4 - \frac{1}{2}x_6
$$

We repeat our pivot process and choose $x_4$ to enter but there is no leaving variable and so the LP is unbounded. We read off the solutions $(1 + \frac{1}{3}t, \frac{3}{2}, 0, t, \frac{1}{2}, 0, 0)^T$ for $t \geq 0$ with $z = \frac{7}{2} + \frac{1}{3}t$. When you have found an LP is unbounded you must give such a parametric solution demonstrating that it is unbounded.