MATH 340 Pivots Preserve the Set of All Solutions

It is crucial that we know that the pivot process preserves the set of solutions to the dictionary. We will not concern ourselves with positivity. You should be able to see that the result applies to all positive solutions as well. Consider a general dictionary with the indices reordered so that the current basis is \( x_{n+1}, x_{n+2}, \ldots, x_{n+m} \):

\[
\begin{align*}
x_{n+i} &= b_i - \sum_{j=1}^{n} a_{ij} x_j & i = 1, 2, \ldots, m \\
z &= \text{const} + \sum_{j=1}^{n} c_j x_j
\end{align*}
\]

(1)

Consider a pivot where \( x_k \) enters and \( x_{n+l} \) leaves. For this we would normally have \( c_k > 0 \) and \( a_{kl} > 0 \) but this is irrelevant for what follows. We must have \( a_{kl} \neq 0 \) for the pivot to make sense.

\[
\begin{align*}
x_k &= \frac{1}{a_{lk}} (b_l - \sum_{j=1, j \neq k}^{n} a_{lj} x_j - x_{n+l}) \\
x_{n+i} &= b_i - \sum_{j=1, j \neq k}^{n} a_{ij} x_j - \frac{a_{ik}}{a_{lk}} (b_l - \sum_{j=1, j \neq k}^{n} a_{lj} x_j - x_{n+l}) \\
z &= \text{const} + \sum_{j=1, j \neq k}^{n} c_j x_j + \frac{c_k}{a_{lk}} (b_l - \sum_{j=1, j \neq k}^{n} a_{lj} x_j - x_{n+l})
\end{align*}
\]

(2)

A solution to (1) yields a solution to (2) since the equations in (2) are derived from the equations in (1) by linear combinations.

\[
sol^{ns}(1) \subseteq \sol^{ns}(2)
\]

We can now choose to pivot with \( x_{n+l} \) entering and \( x_k \) leaving.

\[
\begin{align*}
x_{n+l} &= b_l - \sum_{j=1}^{n} a_{lj} x_j \\
x_{n+i} &= b_i - \sum_{j=1, j \neq k}^{n} a_{ij} x_j - \frac{a_{ik}}{a_{lk}} (b_l - \sum_{j=1, j \neq k}^{n} a_{lj} x_j - (b_l - \sum_{j=1}^{n} a_{lj} x_j)) \\
&= b_i - \sum_{j=1, j \neq k}^{n} a_{ij} x_j - \frac{a_{ik}}{a_{lk}} (a_{lk} x_k) \\
&= b_i - \sum_{j=1}^{n} a_{ij} x_j & i = 1, 2, \ldots, m, i \neq l \\
z &= \text{const} + \sum_{j=1, j \neq k}^{n} c_j x_j + \frac{c_k}{a_{lk}} (b_l - \sum_{j=1, j \neq k}^{n} a_{lj} x_j - (b_l - \sum_{j=1}^{n} a_{lj} x_j)) \\
&= \text{const} + \sum_{j=1, j \neq k}^{n} c_j x_j + \frac{c_k}{a_{lk}} (a_{lk} x_k) \\
&= \text{const} + \sum_{j=1}^{n} c_j x_j
\end{align*}
\]

(3)

We have obtained a new dictionary (3) that is identical to (1)! But as before

\[
sol^{ns}(2) \subseteq \sol^{ns}(3)
\]

We conclude using \( \sol^{ns}(1) = \sol^{ns}(3) \) that

\[
\sol^{ns}(1) = \sol^{ns}(2)
\]

and so the pivot operation preserves solutions.