MATH 340 Second Example of pivoting to optimal solution(s). Richard Anstee

In general optimal solutions are not unique (although of course the optimal value of the objective function z would be unique!). Consider the following minor variant of our previous problem where we have increased the coefficient of x_2 in the objective function from 3 to 5.

$$\begin{array}{rrrrrcl} \max & 4x_1 & +5x_2 & +x_3 & +x_4 & & \\ & x_1 & +2x_2 & & -x_4 & \leq & 3 \\ & 2x_1 & +x_2 & -x_3 & +x_4 & \leq & 2 \\ & & x_2 & +x_3 & & \leq & 2 \end{array} \quad x_1, x_2, x_3, x_4 \geq 0$$

We add slack variables x_5, x_6, x_7 corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables $x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0$. We form our first dictionary

$$\begin{array}{rcrrrrr} x_5 & = & 3 & -x_1 & -2x_2 & & +x_4 \\ x_6 & = & 2 & -2x_1 & -x_2 & +x_3 & -x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & & 4x_1 & +5x_2 & +x_3 & +x_4 \end{array}$$

Now ignore Anstee's rule and have x_1 enter and x_6 leave which yields the dictionary

$$\begin{array}{rcrrrrr} x_5 & = & 2 & +\frac{1}{2}x_6 & -\frac{3}{2}x_2 & -\frac{1}{2}x_3 & +\frac{3}{2}x_4 \\ x_1 & = & 1 & -\frac{1}{2}x_6 & -\frac{1}{2}x_2 & +\frac{1}{2}x_3 & -\frac{1}{2}x_4 \\ x_7 & = & 2 & & -x_2 & -x_3 & \\ z & = & 4 & -2x_6 & +3x_2 & +3x_3 & -x_4 \end{array}$$

Again ignore Anstee's rule and have x_3 enter and x_7 leave to yield the following dictionary.

$$\begin{array}{rcrrrrr} x_5 & = & 1 & +\frac{1}{2}x_6 & -x_2 & +\frac{1}{2}x_7 & +\frac{3}{2}x_4 \\ x_1 & = & 2 & -\frac{1}{2}x_6 & -x_2 & -\frac{1}{2}x_7 & -\frac{1}{2}x_4 \\ x_3 & = & 2 & & -x_2 & -x_7 & \\ z & = & 10 & -2x_6 & & -3x_7 & -x_4 \end{array}$$

There is an *obvious solution* to these 4 equations, namely $x_5 = 1$, $x_1 = 2$, $x_3 = 2$ and $x_6 = x_2 = x_7 = x_4 = 0$ with $z = 10$. But now the equation $z = 10 - 2x_6 - 3x_7 - x_4$ combined with the three inequalities $x_6 \geq 0$, $x_7 \geq 0$, $x_4 \geq 0$ yields $z \leq 10$. Thus we have found an optimal solution to the new LP. In fact $z \leq 10$ with equality if and only if $x_6 = x_7 = x_4 = 0$ (the coefficient of x_2 is 0 in the expression for z). Now setting $x_6 = x_7 = x_4 = 0$ in our third dictionary yields

$$\begin{array}{rcrr} x_5 & = & 1 & -x_2 \\ x_1 & = & 2 & -x_2 \\ x_3 & = & 2 & -x_2 \\ z & = & 10 & \end{array}$$

We deduce that $0 \leq x_2 \leq 1$ and setting $t = x_2$ we can write all possible optimal solutions to the LP as $x_1 = 2 - t$, $x_2 = t$, $x_3 = 2 - t$, $x_4 = 0$ with $x_5 = 1 - t$, $x_6 = 0$ and $x_7 = 0$ and $z = 10$. Remember that you can check this.

Having pivoted through several dictionaries, it is interesting to see how the Revised Simplex

Formulas apply. Consider the final dictionary.

$$\begin{array}{rcrrrrr} x_5 & = & 1 & +\frac{1}{2}x_6 & -x_2 & +\frac{1}{2}x_7 & +\frac{3}{2}x_4 \\ x_1 & = & 2 & -\frac{1}{2}x_6 & -x_2 & -\frac{1}{2}x_7 & -\frac{1}{2}x_4 \\ x_3 & = & 2 & & -x_2 & -x_7 & \\ z & = & 10 & -2x_6 & & -3x_7 & -x_4 \end{array}$$

Recall that $[A\,I]$ is

$$\begin{array}{ccccccc} x_1 & x_2 & x_3 & x_4 & x_5 & x_6 & x_7 \\ 1 & 2 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}$$

The basic variables are x_5, x_1, x_3 and so

$$B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 1 \end{bmatrix}$$

We compute (Wolfram? Matlab?)

$$B^{-1} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix}$$

When we do the Revised Simplex Method, we will label the rows ad columns of B and B^{-1} since obviously row and column order matter when multiplying matrices. Imagine the non basic variables in the order x_2, x_4, x_6, x_7.

$$B^{-1}A_N = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 1 \end{bmatrix}$$

The minus sign is set to trip you up! Then

$$-B^{-1}A_n\mathbf{x}_N = \begin{bmatrix} 1 & -\frac{3}{2} & -\frac{1}{2} & -\frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 1 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_2 \\ x_4 \\ x_6 \\ x_7 \end{bmatrix} = - \begin{bmatrix} x_2 & -\frac{3}{2}x_4 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 \\ x_2 & +\frac{1}{2}x_4 & +\frac{1}{2}x_6 & +\frac{1}{2}x_7 \\ x_2 & & & +x_7 \end{bmatrix}$$

$$= \begin{bmatrix} -x_2 & +\frac{3}{2}x_4 & +\frac{1}{2}x_6 & +\frac{1}{2}x_7 \\ -x_2 & -\frac{1}{2}x_4 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 \\ -x_2 & & & -x_7 \end{bmatrix}$$

While these are not in the variable order of the dictionary above we can see that they are the same.

$$B^{-1}\mathbf{b} = \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & \frac{1}{2} \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 2 \end{bmatrix}$$

Thus

$$B^{-1}\mathbf{b} - B^{-1}A_N\mathbf{x}_N = \begin{bmatrix} 1 & -x_2 & +\frac{3}{2}x_4 & +\frac{1}{2}x_6 & +\frac{1}{2}x_7 \\ 2 & -x_2 & -\frac{1}{2}x_4 & -\frac{1}{2}x_6 & -\frac{1}{2}x_7 \\ 2 & -x_2 & & & -x_7 \end{bmatrix}$$

which matches the final dictionary above.