Example of pivoting to optimal solution(s).

We considered the following LP in standard inequality form

\[
\begin{align*}
\text{max} & \quad 4x_1 + 3x_2 + x_3 + x_4 \\
& \quad x_1 + 2x_2 - x_4 \leq 3 \quad x_1, x_2, x_3, x_4 \geq 0 \\
& \quad 2x_1 + x_2 - x_3 + x_4 \leq 2 \\
& \quad x_2 + x_3 \leq 2
\end{align*}
\]

We add slack variables \(x_5, x_6, x_7\) corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables \(x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0\). We form our first dictionary

\[
\begin{align*}
x_5 &= 3 -x_1 -2x_2 + x_4 \\
x_6 &= 2 -2x_1 -x_2 + x_3 - x_4 \\
x_7 &= 2 -x_2 -x_3 \\
z &= 4x_1 + 3x_2 + x_3 + x_4
\end{align*}
\]

It is traditional to use \(z\) for the objective function. There is an obvious solution to these 4 equations, namely \(x_5 = 3, x_6 = 2, x_7 = 2\) and \(x_1 = x_2 = x_3 = x_4 = 0\) with \(z = 0\). (this is called a basic feasible solution)

We now use Anstees rule trying to increase a variable from 0 in the current obvious solution so we greedily choose \(x_1\) to increase and hence enter. We leave \(x_2 = x_3 = x_4 = 0\). The choice of \(x_1\) as the variable with the largest coefficient in dictionary expression for \(z\) (and in the case of ties choosing the variable of smallest subscript) is called Anstees Rule in this course.

\[
\begin{align*}
x_5 &= 3 -x_1 \\
x_6 &= 2 -2x_1 \\
x_7 &= 2 \\
z &= 4x_1
\end{align*}
\]

we deduce that \(x_1\) can be increased to 1 while decreasing \(x_6\) to 0. We obtain a new dictionary by having \(x_1\) only appear on the left and \(x_6\) is now on the right of the equation signs.

\[
\begin{align*}
x_5 &= 2 +\frac{1}{2}x_6 -\frac{3}{2}x_2 -\frac{1}{2}x_3 +\frac{3}{2}x_4 \\
x_1 &= 1 -\frac{1}{2}x_6 -\frac{1}{2}x_2 +\frac{1}{2}x_3 -\frac{1}{2}x_4 \\
x_7 &= 2 -x_2 -x_3 \\
z &= 4 -2x_6 +x_2 +3x_3 - x_4
\end{align*}
\]

There is an obvious solution to these 4 equations, namely \(x_5 = 2, x_1 = 1, x_7 = 2\) and \(x_6 = x_2 = x_3 = x_4 = 0\) with \(z = 4\). Note how I keep all the entries of each variable in neat columns. It makes adding and subtracting equations much more reliable.

By Anstees rule we would wish to increase \(x_3\) leaving \(x_6 = x_2 = x_4 = 0\)

\[
\begin{align*}
x_5 &= 2 -\frac{1}{2}x_3 \\
x_1 &= 1 +\frac{1}{2}x_3 \\
x_7 &= 2 -x_3 \\
z &= 4 +3x_3
\end{align*}
\]

and we deduce that we could increase \(x_3\) to 2 while driving \(x_7\) to 0 and so we say \(x_3\) enters and \(x_7\) leaves.
\[ x_5 = 1 + \frac{1}{2}x_6 - x_2 + \frac{1}{2}x_7 + \frac{3}{2}x_4 \\
\]
\[ x_1 = 2 - \frac{1}{2}x_6 - x_2 - \frac{3}{2}x_7 - \frac{1}{2}x_4 \\
\]
\[ x_3 = 2 - x_2 - x_7 \\
\]
\[ z = 10 - 2x_6 - 2x_2 - 3x_7 - x_4 \]

There is an obvious solution to these 4 equations, namely \( x_5 = 1, x_1 = 2, x_3 = 2 \) and \( x_6 = x_2 = x_7 = x_4 = 0 \) with \( z = 10 \). But now the equation \( z = 10 - 2x_6 - 2x_2 - 3x_7 - x_4 \) combined with the four inequalities \( x_6 \geq 0, x_2 \geq 0, x_7 \geq 0, x_4 \geq 0 \) yields \( z \leq 10 \). Thus we have found an optimal solution to the LP. In fact \( z \leq 10 \) with equality if and only if \( x_6 = x_2 = x_7 = x_4 = 0 \). Now setting \( x_6 = x_2 = x_7 = x_4 = 0 \) in our third dictionary yields \( x_5 = 1, x_1 = 2, x_3 = 2 \). Thus we have found the unique optimal solution in this case.

In general optimal solutions are not unique (although of course the optimal value of the objective function \( z \) would be unique!). Consider the following minor variant of our problem where we have increased the coefficient of \( x_2 \) in the objective function from 3 to 5.

\[
\text{max } 4x_1 + 5x_2 + x_3 + x_4 \leq \\
x_1 + 2x_2 - x_4 \leq 3 \\
2x_1 + x_2 - x_3 + x_4 \leq 2 \\
x_2 + x_3 \leq 2
\]

We add slack variables \( x_5, x_6, x_7 \) corresponding to the difference between the left and right hand sides of the three constraints so that all 7 variables \( x_1, x_2, x_3, x_4, x_5, x_6, x_7 \geq 0 \). We form our first dictionary

\[
x_5 = 3 - 2x_1 - 2x_2 + x_4 \\
x_6 = 2 - 2x_1 - x_2 + x_3 - x_4 \\
x_7 = 2 - x_2 - x_3 \\
z = 4x_1 + 5x_2 + x_3 + x_4
\]

Now ignore Anstees rule and have \( x_1 \) enter and \( x_6 \) leave which yields the dictionary

\[
x_5 = 2 + \frac{1}{2}x_6 - \frac{3}{2}x_2 - \frac{1}{2}x_3 + \frac{3}{2}x_4 \\
x_1 = 1 - \frac{1}{2}x_6 - \frac{1}{2}x_2 + \frac{1}{2}x_3 - \frac{1}{2}x_4 \\
x_7 = 2 - x_2 - x_3 \\
z = 4 - 2x_6 + 3x_2 + 3x_3 - x_4
\]

Again ignore Anstees rule and have \( x_3 \) enter and \( x_7 \) leave to yield the following dictionary.

\[
x_5 = 1 + \frac{1}{2}x_6 - x_2 + \frac{1}{2}x_7 + \frac{3}{2}x_4 \\
x_1 = 2 - \frac{1}{2}x_6 - x_2 - \frac{3}{2}x_7 - \frac{1}{2}x_4 \\
x_3 = 2 - x_2 - x_7 \\
z = 10 - 2x_6 - 3x_7 - x_4
\]

There is an obvious solution to these 4 equations, namely \( x_5 = 1, x_1 = 2, x_3 = 2 \) and \( x_6 = x_2 = x_7 = x_4 = 0 \) with \( z = 10 \). But now the equation \( z = 10 - 2x_6 - 3x_7 - x_4 \) combined with the three inequalities \( x_6 \geq 0, x_7 \geq 0, x_4 \geq 0 \) yields \( z \leq 10 \). Thus we have found an optimal solution to the new LP. In fact \( z \leq 10 \) with equality if and only if \( x_6 = x_7 = x_4 = 0 \) (the coefficient of \( x_2 \) is 0 in the expression for \( z \)). Now setting \( x_6 = x_7 = x_4 = 0 \) in our third dictionary yields

\[
x_5 = 1 - x_2 \\
x_1 = 2 - x_2 \\
x_3 = 2 - x_2 \\
z = 10
\]
We deduce that $0 \leq x_2 \leq 1$ and setting $t = x_2$ we can write all possible optimal solutions to the LP as $x_1 = 2 - t$, $x_2 = t$, $x_3 = 2 - t$ (I made an error here on the boards, sorry), $x_4 = 0$ with $x_5 = 1 - t$, $x_6 = 0$ and $x_7 = 0$ and $z = 10$. 